BIVARIATE VECTOR AUTOREGRESSIVE SYSTEMS

In this note, we consider some of the forms of dynamic equations that are associated with a bivariate vector autoregressive process.

The basic model takes the form of

$$y(t) = \sum_{i=1}^{p} \phi_i y(t-i) + \sum_{i=1}^{p} \beta_i x(t-i) + \varepsilon(t), \tag{1}$$

$$x(t) = \sum_{i=1}^{p} \psi_i x(t-i) + \sum_{i=1}^{p} \delta_i y(t-i) + \eta(t). \tag{2}$$

These equations can be written in a more summary notation which uses polynomials in the lag operator to represent the various sums. Thus

$$\phi(L)y(t) - \beta(L)x(t) = \varepsilon(t), \tag{3}$$

$$-\delta(L)y(t) + \psi(L)x(t) = \eta(t), \tag{4}$$

where $\phi(L) = 1 - \phi_1 L - \cdots - \phi_p L^p$, $\beta(L) = \beta_1 L + \cdots + \beta_p L^p$, $\psi(L) = 1 - \psi_1 L - \cdots - \psi_p L^p$ and $\delta(L) = \delta_1 L + \cdots + \delta_p L^p$.

The motive power that drives this dynamic system comes from the disturbance sequences $\varepsilon(t)$ and $\eta(t)$, and it is interesting to assess their relative strengths and to determine their impacts upon the two equations. Such matters can be investigated in terms of a several alternative representations of the bivariate system.

To begin, consider representing the two equations in their rational forms, which are

$$y(t) = \frac{\beta(L)}{\phi(L)}x(t) + \frac{1}{\phi(L)}\varepsilon(t), \tag{5}$$

$$x(t) = \frac{\delta(L)}{\psi(L)}y(t) + \frac{1}{\psi(L)}\eta(t). \tag{6}$$

Here, the coupling of the two equations ensures that the terms on the RHS are mutually correlated. Therefore the variances of $y(t)$ and $x(t)$ cannot be partitioned directly amongst their component parts.

An alternative approach is to express $y(t)$ and $x(t)$ solely in terms of the two innovations sequences $\varepsilon(t)$ and $\eta(t)$. This is done inverting the relationship in (5) and (6) to give

$$y(t) = \frac{\psi(L)}{\pi(L)}\varepsilon(t) + \frac{\beta(L)}{\pi(L)}\eta(t), \tag{7}$$

$$x(t) = \frac{\delta(L)}{\pi(L)}\varepsilon(t) + \frac{\phi(L)}{\pi(L)}\eta(t), \tag{8}$$

where $\pi(L) = \phi(L)\psi(L) - \beta(L)\delta(L)$. In this representation, the two equations have been decoupled in the sense that there is no longer any feedback from one to the other. The components on the RHS of the equations are in the form of ARMA processes; and the stability of the sytem as a whole depends upon the stability of the constituent processes. The necessary and sufficient condition for stability is that the roots of $\pi(z) = 0$ should lie outside the unit circle.

A problem with this representation is that there is liable to be a degree of contemporaneous correlation between innovations sequences; and, therefore, the variance of the observable sequences $y(t)$ and $x(t)$ will still not equal the sum of the variances of the components on the RHS.

The problem can be overcome by transforming the equations so that the innovations sequences do become uncorrelated. Consider the innovation sequence $\eta(t)$ within the context of equation (7) which is for $y(t)$. We may decompose $\eta(t)$ into a component which is in the space spanned by $\varepsilon(t)$ and a component $\zeta(t)$ which is in the orthogonal complement of the space. Thus

$$\text{(9)} \qquad \begin{aligned} \eta(t) &= \frac{\sigma_{\eta\varepsilon}}{\sigma_\varepsilon^2}\varepsilon(t) + \left\{\eta(t) - \frac{\sigma_{\eta\varepsilon}}{\sigma_\varepsilon^2}\varepsilon(t)\right\} \\ &= \frac{\sigma_{\eta\varepsilon}}{\sigma_\varepsilon^2}\varepsilon(t) + \zeta(t), \end{aligned}$$

where $\sigma_\varepsilon^2 = V\{\varepsilon(t)\}$ and $\sigma_{\varepsilon\eta}^2 = V\{\varepsilon(t), \eta(t)\}$. Substituting (9) in equation (7) and gathering the terms in $\varepsilon(t)$ gives

$$\text{(9)} \qquad y(t) = \frac{\alpha(L)}{\pi(L)}\varepsilon(t) + \frac{\beta(L)}{\pi(L)}\zeta(t),$$

where

$$\text{(10)} \qquad \alpha(L) = \psi(L) + \frac{\sigma_{\eta\varepsilon}}{\sigma_\varepsilon^2}\beta(L).$$

By a similar reparametrisation, the equation in $x(t)$ becomes

$$\text{(11)} \qquad x(t) = \frac{\gamma(L)}{\pi(L)}\eta(t) + \frac{\delta(L)}{\pi(L)}\xi(t)$$

where

$$\text{(12)} \qquad \begin{aligned} \gamma(L) &= \phi(L) + \frac{\sigma_{\eta\varepsilon}}{\sigma_\eta^2}\delta(L), \\ \xi(t) &= \varepsilon(t) - \frac{\sigma_{\eta\varepsilon}}{\sigma_\eta^2}\eta(t), \end{aligned}$$

and where $\eta(t)$ and $\xi(t)$ are mutually uncorrelated.