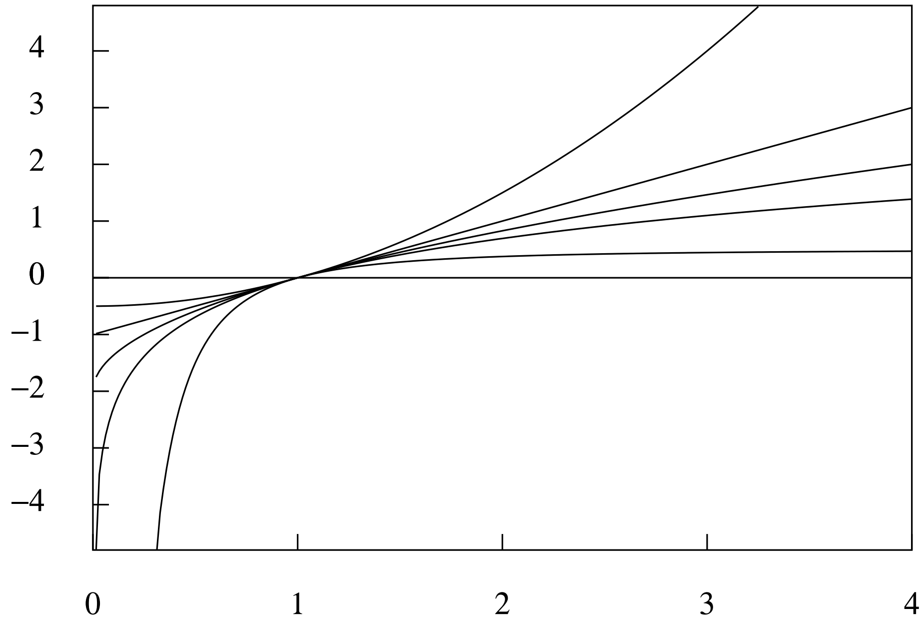

Figure 1. The Box–Cox transformation for $\lambda = -2.0, -1.0, 0.0, 1.0, 2.0$.

THE BOX–COX TRANSFORMATION

The Box–Cox transformation is a modified power transformation that is defined by

$$(1) \qquad B(x,\lambda) = \begin{cases} \dfrac{x^\lambda - 1}{\lambda}, & \text{if } \lambda \neq 0, \\ \ln(x), & \text{if } \lambda = 0. \end{cases}$$

The argument x is strictly positive. The value of the function at $\lambda = 0$ is established using L'Hopital's rule whereby, if $f(\lambda)$ and $g(\lambda)$ are two differentiable functions, then

$$(2) \qquad \lim(\lambda \to a)\{f(\lambda)/g(\lambda)\} = \lim(\lambda \to a)\{f'(\lambda)/g'(\lambda)\}.$$

The value of the function $B(x,\lambda)$ at $\lambda = 1$ is $x - 1$.

To apply L'Hopital's rule, we set $x^\lambda = e^q$, which gives $q = \lambda \ln(x)$. From this, we have $d(x^\lambda)/d\lambda = x^\lambda \ln(x)$, within which $\lim(\lambda \to 0)x^\lambda = 1$. The derivative of the denominator of $B(x,\lambda)$ with respect to λ is unity.

Values of λ outside the interval $[0,1]$ are allowable. When $\lambda > 1$, the slope of $B(x,\lambda)$ increases with x, whereas, when $\lambda < 1$, it decreases with x. When $\lambda < 0$, there is $B(x,\lambda) = |\lambda|^{-1}(1 - x^{-|\lambda|})$. This tends to $-\infty$ as $x \to 0$ and to an upper asymptote of $|\lambda|^{-1}$ as $x \to \infty$.

The Box–Cox transformation can be used in the context of a regression model when it is uncertain whether an explanatory variable x should be present in its original form or as a logarithmic. If the function $B(x,\lambda)$ is put in place of the variable, then the value of λ can be estimated in the company of the regression parameters; and a statistical test can be used to determine whether there should be $\lambda = 0$ or $\lambda = 1$. In the latter case, the variable will be present in the form of $x - 1$, but the unit can be absorbed into the intercept term of the regression equation.

The transformation is sometimes applied to time series, in which the variability increases with the level, in an attempt to reduce the series to a state of homogeneity. It is generally presupposed that the values of the series are non-negative. In particular, for positive data with a standard deviation that increases linearly with the level, the variance can be stabilised by choosing $\lambda = 0$, which results in a logarithmic transformation.

The result can be understood by considering the case of a variable with a mean trajectory $\bar{y}(t) = y_0 e^{rt}$, which fluctuates within the widening interval $(y_L e^{rt}, y_U e^{rt})$. The width of the interval is a constant fraction of $\bar{y}(t)$. When logs are taken, the exponential mean trajectory becomes a linear one and the interval becomes constant.