THE BEVERIDGE–NELSON DECOMPOSITION

In this note, we reproduce the algebra of the paper of Beveridge and Nelson (1981) precisely in the manner in which it was originally presented. Then, we show how it corresponds to a simple instance of the ordinary remainder theorem of polynomial algebra and to an instance of a partial-fraction decomposition of a rational function.

Beveridge and Nelson write the first differences of an ARIMA process with drift as

$$y_t - y_{t-1} = w_t = \mu + \{\varepsilon_t + \lambda_1\varepsilon_{t-1} + \cdots\} \tag{1}$$

Here, y_t, which they denote by z_t, stands for the level of the variable, whilst w_t is its first difference. The drift parameter is μ.

The expectation of y_{t+k}, conditional on the data available at time t, is

$$E(y_{t+k}|\mathcal{I}_t) = \hat{y}_{t+k|t} = y_t + \{\hat{w}_{t+1|t} + \hat{w}_{t+2|t} + \cdots + \hat{w}_{t+k|t}\}. \tag{2}$$

Within this expression, there is a sequence of expectations conditional on the information in $\mathcal{I}_t$, which are $E(w_{t+j}|\mathcal{I}_t) = \hat{w}_{t+j|t}; j = 1, \ldots, k$:

$$\begin{aligned}
\hat{w}_{t+1|t} &= \mu + \{\lambda_1\varepsilon_t + \lambda_2\varepsilon_{t-1} + \lambda_3\varepsilon_{t-2} + \cdots\},\\
\hat{w}_{t+2|t} &= \mu + \{\lambda_2\varepsilon_t + \lambda_3\varepsilon_{t-1} + \lambda_4\varepsilon_{t-2} + \cdots\},\\
\hat{w}_{t+3|t} &= \mu + \{\lambda_3\varepsilon_t + \lambda_4\varepsilon_{t-1} + \lambda_5\varepsilon_{t-2} + \cdots\},\\
&\vdots\\
\hat{w}_{t+k|t} &= \mu + \{\lambda_k\varepsilon_t + \lambda_{k+1}\varepsilon_{t-1} + \lambda_{k+2}\varepsilon_{t-2} + \cdots\}.
\end{aligned} \tag{3}$$

Substituting these into (2) gives

$$\begin{aligned}
\hat{y}_{t+k|t} &= y_t + \{\hat{w}_{t+1|t} + \hat{w}_{t+2|t} + \cdots + \hat{w}_{t+k|t}\}\\
&= y_t + k\mu + \left(\sum_{j=1}^{k}\lambda_i\right)\varepsilon_t + \left(\sum_{j=2}^{k+1}\lambda_i\right)\varepsilon_{t-1} + \left(\sum_{j=3}^{k+2}\lambda_i\right)\varepsilon_{t-2} + \cdots
\end{aligned} \tag{4}$$

For very large values of the lead time k, there is

$$\hat{y}_{t+k|t} \simeq y_t + k\mu + \left(\sum_{j=1}^{\infty}\lambda_i\right)\varepsilon_t + \left(\sum_{j=2}^{\infty}\lambda_i\right)\varepsilon_{t-1} + \left(\sum_{j=3}^{\infty}\lambda_i\right)\varepsilon_{t-2} + \cdots \tag{5}$$

The forecast function represents a linear extrapolation from a point, determined at time t, that can be regarded as the intercept of the function. The slope is μ and the intercept is

$$\bar{y}_t = y_t + \left(\sum_{j=1}^{\infty}\lambda_j\right)\varepsilon_t + \left(\sum_{j=2}^{\infty}\lambda_j\right)\varepsilon_{t-1} + \left(\sum_{j=3}^{\infty}\lambda_j\right)\varepsilon_{t-2} + \cdots \tag{6}$$

Taking first differences of (6) gives

$$\bar{y}_t - \bar{y}_{t-1} = w_t + \left(\sum_{j=1}^{\infty} \lambda_j\right) \varepsilon_t - (\lambda_1 \varepsilon_{t-1} + \lambda_2 \varepsilon_{t-2} + \cdots). \tag{7}$$

Using equation (1), we can reduce (6) to

$$\bar{y}_t - \bar{y}_{t-1} = \mu + \left(\sum_{j=0}^{\infty} \lambda_j\right) \varepsilon_t. \tag{8}$$

This indicates that the intercept $\bar{y}_t$ is determined by a pure first-order random walk with drift. Returning to equation (6), we see that we can write

$$y_t = \bar{y}_t + \{\gamma_0 \varepsilon_t + \gamma_1 \varepsilon_{t-1} + \cdots\}, \quad \text{where} \quad \gamma_j = -\sum_{i=j}^{\infty} \lambda_i. \tag{9}$$

This expresses y_t as the sum of a random walk, $\bar{y}_t$, based on an accumulation of the elements $\{\varepsilon_{t-j}; j = 0, 1, 2, \ldots\}$ of a white-noise process, and a stationary stochastic process, driven by the same white noise.

At the heart of the Beveridge–Nelson decomposition is the fact that any polynomial or rational function, which we can denote by $\lambda(z) = \theta(L)/\phi(L)$, can be written in the form of

$$\lambda(z) = \lambda(1) + (1 - z)\gamma(z). \tag{10}$$

Here, $\gamma(z)$ is the quotient of the division of $\lambda(z)$ by $(1 - z)$. The remainder of the division is $\delta = \lambda(1)$. This is obtained by setting $z = 1$ in the formula $\lambda(z) = (1 - z)\gamma(z) + \delta$. In the case of a rational function with the factor $1 - z$ in the denominator, we have the following partial-fraction expansion:

$$\frac{\lambda(z)}{1 - z} = \frac{\theta(z)}{\phi(z)(1 - z)} = \frac{\rho(z)}{\phi(z)} + \frac{\delta}{1 - z}. \tag{11}$$

Multiplying both sides by $1 - z$ and setting $z = 1$ gives $\delta = \theta(1)/\phi(1) = \lambda(1)$, which is the same result as before.

Now consider consider the process postulated by Beveridge and Nelson:

$$\begin{aligned} \nabla y(t) &= \mu + \frac{\theta(L)}{\phi(L)} \varepsilon(t) \\ &= \mu + \lambda(L)\varepsilon(t). \end{aligned} \tag{12}$$

The final equation of the RHS is equation (1) in an alternative notation. The Beveridge–Nelson decomposition is as follows:

$$\begin{aligned}\nabla y(t) &= \mu + \lambda(L)\varepsilon(t)\\ &= \mu + \lambda(1)\varepsilon(t) + \nabla\gamma(L)\varepsilon(t),\end{aligned} \tag{13}$$

where $\nabla = 1 - L$. This uses only equation (10). Using the complex argument z in place of the lag operator L and $\nabla(z) = 1 - z$ in place of $\nabla = 1 - L$, and placing z within the radius of convergence, will allow us to use equation (11) to write

$$\begin{aligned}y(z) &= \frac{\mu}{\nabla(z)} + \frac{\lambda(z)}{\nabla(z)}\varepsilon(z)\\ &= \frac{\mu}{1-z} + \frac{\theta(1)}{\phi(1)}\frac{\varepsilon(z)}{(1-z)} + \frac{\rho(z)}{\phi(z)}\varepsilon(z).\end{aligned} \tag{14}$$

It remains to determine the parameters of the series expansion of the rational function $\gamma(z) = \rho(z)/\phi(z)$, which Beveridge and Nelson use in their exposition. These can be obtained by equating the coefficients associated with same powers of z from both sides of the following equation, which is an explicit rendering of equation (10):

$$\{\lambda_0 + \lambda_1 z + \lambda_2 z^2 + \cdots\} = \delta + (1-z)\{\gamma_0 + \gamma_1 z + \gamma_2 z^2 + \cdots\}. \tag{15}$$

The results are

$$\begin{aligned}\lambda_0 &= \delta + \gamma_0, & \gamma_0 &= \lambda_0 - \delta,\\ \lambda_1 &= \gamma_1 - \gamma_0, & \gamma_1 &= \lambda_1 + \gamma_0,\\ \lambda_2 &= \gamma_2 - \gamma_1, & \gamma_2 &= \lambda_2 + \gamma_1,\\ \lambda_3 &= \gamma_3 - \gamma_2, & \gamma_3 &= \lambda_3 + \gamma_2.\end{aligned} \tag{16}$$

But, from the remainder theorem, $\delta = \{\lambda_0 + \lambda_1 + \lambda_2 + \cdots\}$, so we have

$$\begin{aligned}\gamma_0 &= -(\lambda_1 + \lambda_2 + \lambda_3 + \cdots),\\ \gamma_1 &= -(\lambda_2 + \lambda_3 + \lambda_4 + \cdots),\\ \gamma_2 &= -(\lambda_3 + \lambda_4 + \lambda_5 + \cdots),\\ \gamma_3 &= -(\lambda_4 + \lambda_5 + \lambda_6 + \cdots).\end{aligned} \tag{17}$$

Reference

Beveridge, S., and C.R. Nelson, (1981), A New Approach to the Decomposition of Economic Time Series into Permanent and Transitory Components with Particular Attention to Measurement of the Business Cycle, *Journal of Monetary Economics 7,* 151–172.

We might also wish to calculate the coefficients of the polynomial $\rho(z)$ of (14), which represents the numerator of the stable transfer function in the Beveridge–Nelson decomposition. Observe that equation (11) implies that

$$\begin{aligned}\frac{\rho(z)}{\phi(z)} &= \frac{\rho(z)}{\phi(z)\nabla(z)} - \frac{\theta(1)/\phi(1)}{\nabla(z)} \\ &= \frac{\theta(z) - \{\theta(1)/\phi(1)\}\phi(z)}{\phi(z)\nabla(z)} = \frac{\beta(z)}{\phi(z)\nabla(z)}.\end{aligned} \tag{18}$$

Setting $z = 1$ in $\beta(z) = \theta(z) - \{\theta(1)/\phi(1)\}\phi(z)$ gives $\beta(1) = 0$. Therefore, $z = 1$ is a root of the equation $\beta(z) = 0$. That is to say, $\beta(z)$ contains the factor $\nabla(z) = 1 - z$, which divides the polynomial exactly. A further implication is that $\beta_0 + \beta_1 + \cdots + \beta_r = 0$, where r is the degree of the $\beta(z)$.

To find the coefficients of $\rho(z)$, we equate the coefficients of the same powers of z on both sides of the equation $\beta(z) = \rho(z)(1 - z)$. The result is

$$\begin{aligned} \beta_0 &= \rho_0, & \rho_0 &= \beta_0, \\ \beta_1 &= \rho_1 - \rho_0, & \rho_1 &= \beta_1 + \rho_0 = \beta_0 + \beta_1, \\ \beta_2 &= \rho_2 - \rho_1, & \rho_2 &= \beta_2 + \rho_1 = \beta_0 + \beta_1 + \beta_2, \\ &\vdots & &\vdots \\ \beta_{r-1} &= \rho_{r-1} - \rho_{r-2}, & \rho_{r-1} &= \beta_{r-1} + \rho_{r-2} = \beta_0 + \cdots + \beta_{r-1}. \\ \beta_r &= -\rho_{r-1}, \end{aligned} \tag{19}$$