REPARAMETRISATION OF DYNAMIC MODELS

In this note, we demonstrate a simple identity affecting polynomials in the lag operator and we show how this can be used in describing relationships between cointegrated time series. The same identity has been used to express an ARIMA process as the sum of a stationary stochastic process and an ordinary random walk. This expression is commonly know as the Beveridge–Nelson decomposition after its original proponents.

A Polynomial Identity

Let $\beta(z) = \beta_0 + \beta_1 z + \cdots + \beta_k z^k = \sum_{j=0}^{k} \beta_j z^j$ be a polynomial of degree k in the argument z. We wish to show that this can be written in the following forms:

$$\begin{aligned}\beta(z) &= \beta(1) + \nabla(z)\gamma(z) \\ &= z^n\beta(1) + \nabla(z)\delta_n(z),\end{aligned} \tag{1}$$

where $\nabla(z) = 1 - z$, where $0 \leq n \leq k$ and where $\gamma(z)$ and $\delta_n(z)$ are both polynomials of degree $k-1$. Also, $\beta(1)$ is the constant which is obtained by setting $z = 1$ in the polynomial $\beta(z)$.

To obtain the first expression on the RHS of (1), we divide $\beta(z)$ by $\nabla(z) = 1 - z$ to obtain a quotient of $\gamma(z)$ and a remainder of δ:

$$\beta(z) = \gamma(z)(1 - z) + \delta. \tag{2}$$

Setting $z = 1$ in this equation gives

$$\delta = \beta(1) = \beta_0 + \beta_1 + \cdots + \beta_k. \tag{3}$$

This is an instance of the well-known remainder theorem of polynomial division. The coefficients of the quotient polynomial $\gamma(z)$ are given by

$$\gamma_p = -\sum_{j=p+1}^{k} \beta_j, \quad \text{where} \quad p = 0, \ldots, k-1. \tag{4}$$

There is a wide variety of ways in which these coefficients may de derived, including the familiar method of long division. Probably, the easiest way is via the method of synthetic division which may be illustrated by an example.

Example. Consider the case where $k = 3$. Then

$$\beta_0 + \beta_1 z + \beta_2 z^2 + \beta_3 z^3 = (\gamma_0 + \gamma_1 z + \gamma_2 z^2)(1 - z) + \delta. \tag{5}$$

By equating the coefficients associated with the same powers of z on either side of the equation, we obtain the following identities:

$$
\begin{aligned}
\beta_3 &= -\gamma_2 \\
\beta_2 &= -\gamma_1 + \gamma_2 \\
\beta_1 &= -\gamma_0 + \gamma_1 \\
\beta_0 &= \delta + \gamma_0.
\end{aligned} \tag{6}
$$

These can be rearranged to give

$$
\begin{aligned}
\gamma_2 &= -\beta_3 \\
\gamma_1 &= -\beta_2 + \gamma_2 &&= -(\beta_2 + \beta_3) \\
\gamma_0 &= -\beta_1 + \gamma_1 &&= -(\beta_1 + \beta_2 + \beta_3) \\
\delta &= \beta_0 - \gamma_0 &&= \beta_0 + \beta_1 + \beta_2 + \beta_3.
\end{aligned} \tag{7}
$$

To obtain the second expression on the RHS of (1), consider the identity

$$1 = z^n + \nabla(z)(1 + z + \cdots + z^{n-1}), \tag{8}$$

where $1 < n \leq k$. Multiplying both sides by $\beta(1)$ gives

$$\beta(1) = z^n\beta(1) + \nabla(z)\{1 + z + \cdots + z^{n-1}\}\beta(1), \tag{9}$$

On substituting this expression into the first equation of (1) and on defining

$$\delta_n(z) = \gamma(z) + \{1 + z + \cdots + z^{n-1}\}\beta(1), \tag{10}$$

we can write

$$\beta(z) = z^n\beta(1) + \nabla(z)\delta_n(z). \tag{11}$$

This is a general expression which covers both equations of (1), since setting $n = 0$ reduces it to the first equation.

A leading instance is obtained by setting $n = 1$. In that case, equation (9) gives $\beta(1) = z\beta(1) + \nabla(z)\beta(1)$. Substituting this into $\beta(z) = \beta(1) + \nabla(z)\gamma(z)$ gives the following specialisation of equation (11):

$$
\begin{aligned}
\beta(z) &= z\beta(1) + \nabla(z)\{\gamma(z) + \beta(1)\} \\
&= z\beta(1) + \nabla(z)\delta_1(z).
\end{aligned} \tag{12}
$$

By comparing coefficients associated with the same powers of z on both sides of the equation, it can be seen that the constant term of the polynomial $\delta_1(z)$ is just β_0. Reference to (3) and (7) confirms this result.

Reparametrisation of a Distributed Lag Model

Consider a distributed-lag model of the form

$$y(t) = \beta_0 x(t) + \beta_1 x(t-1) + \cdots + \beta_k x(t-k) + \varepsilon(t). \tag{13}$$

This can be written is summary notation as

$$y(t) = \beta(L)x(t) + \varepsilon(t), \tag{14}$$

where $\beta(L) = \beta_0 + \beta_1 L + \cdots + \beta_k L^k$ is a polynomial in the lag operator L.

Using the basic identity from (1), we can set $\beta(L) = \beta(1) + \nabla\delta(L)$. Then taking $y(t-1)$ from both sides of the resulting equation gives

$$\nabla y(t) = \{\beta(1)x(t) - y(t-1)\} + \delta(L)\nabla x(t) + \varepsilon(t), \tag{15}$$

which is an error-correction formulation of equation (14).

Reparametrisation of an Autoregressive Distributed Lag Model

Now consider an equation in the form of

$$y(t) = \phi_1 y(t-1) + \cdots + \phi_p y(t-p) + \beta_0 x(t) + \cdots + \beta_k x(t-k) + \varepsilon(t), \tag{16}$$

which can be written is summary notation as

$$\alpha(L)y(t) = \beta(L)x(t) + \varepsilon(t), \tag{17}$$

with $\alpha(L) = \alpha_0 + \alpha_1 L + \cdots + \alpha_p L^p = 1 - \phi_1 L - \cdots - \phi_p L^p$. On setting $\alpha(L) = \alpha(1)L + \theta_1(L)\nabla$ and $\beta(L) = \beta(1)L + \delta_1(L)\nabla$, this can be rewritten as

$$\{\alpha(1)L + \theta_1(L)\nabla\}y(t) = \{\beta(1)L + \delta_1(L)\nabla\}x(t) + \varepsilon(t), \tag{18}$$

where the leading element of $\theta_1(L)$ is $\alpha_0 = 1$. Define

$$\rho(L) = \rho_1 L + \rho_2 L^2 + \cdots + \rho_p L^p = 1 - \theta_1(L). \tag{19}$$

Then equation (18) can be rearranged to give

$$\begin{aligned}\nabla y(t) &= \{\beta(1)Lx(t) - \alpha(1)Ly(t)\} + \delta_1(L)\nabla x(t) + \rho(L)\nabla y(t) + \varepsilon(t)\\ &= \lambda\{\gamma x(t-1) - y(t-1)\} + \delta_1(L)\nabla x(t) + \rho(L)\nabla y(t) + \varepsilon(t),\end{aligned} \tag{20}$$

where $\lambda = \alpha(1)$ is the so-called adjustment parameter and where $\gamma = \beta(1)/\alpha(1)$ is the steady-state gain of the rational transfer function $\beta(L)/\alpha(L)$. The term $\gamma x(t-1) - y(t-1)$ is described as the equilibrium error; and the value of the error will tend to zero if a steady state is maintained by $x(t)$ and if there are no disturbances. Equation (20) is the classical form of the error-correction equation.

The Beveridge–Nelson Decomposition

Consider an ARIMA model which is represented by the equation

$$\alpha(L)\nabla y(t) = \mu(L)\varepsilon(t) \tag{21}$$

Dividing both sides by $\alpha(L)$ gives

$$\begin{aligned}\nabla y(t) &= \frac{\mu(L)}{\alpha(L)}\varepsilon(t) \\ &= \psi(L)\varepsilon(t),\end{aligned} \tag{22}$$

where $\psi(L)$ stands for the power series expansion of the rational function. If the coefficients of this expansion form an absolutely summable sequence, then $\psi(L) = \psi(1)+\nabla\lambda(L)$, which a decomposition in the form of equation (1). Then

$$\begin{aligned}\nabla y(t) &= \psi(1)\varepsilon(t) + \lambda(L)\nabla\varepsilon(t) \\ &= \nabla v(t) + \nabla w(t).\end{aligned} \tag{23}$$

Here, $v(t) = \psi(1)\nabla^{-1}\varepsilon(t)$ in first term on the RHS stands for a random walk, whereas $w(t) = \lambda(L)\varepsilon(t)$ stands for a stationary stochastic process.

For another perspective on this decomposition, we may consider the following partial-fraction decompostion:

$$\frac{\mu(z)}{\alpha(z)\nabla(z)} = \frac{\gamma(z)}{\alpha(z)} + \frac{\delta}{\nabla(z)}. \tag{24}$$

Multiplying both sides by $\nabla(z)$ gives

$$\frac{\mu(z)}{\alpha(z)} = \frac{\nabla(z)\gamma(z)}{\alpha(z)} + \delta, \tag{25}$$

whence setting $z = 1$ gives $\delta = \mu(1)/\alpha(1)$. Thereafter, we can find $\gamma(z) = \{\mu(z) - \delta\alpha(z)\}/\nabla(z)$.