THE ALMON LAG

A useful way of reducing the number of parameters to be estimated in the distributed-lag model

$$y(t) = \sum_{i=0}^{k} \beta_i x(t-i) + \varepsilon(t) \tag{1}$$

is to assume that the $k+1$ coefficients of $\beta = [\beta_0, \ldots, \beta_k]'$, can be represented by the ordinates of a polynomial $P(i)$ of a degree q which is less than k. Thus it may be specified that

$$\beta_i = P(i) = \sum_{j=0}^{q} \gamma_j i^j \quad \text{for} \quad i = 0, \ldots, k. \tag{2}$$

Let $\Pi = [i^j]$ and let $\gamma = [\gamma_0, \gamma_1, \ldots, \gamma_q]$. Then the equations of equation (2) can be be compiled as $\beta = \Pi\gamma$, which can be written more explicitly as

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 4 & \cdots & 2^q \\ 1 & 3 & 9 & \cdots & 3^q \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & k & k^2 & \cdots & k^q \end{bmatrix} \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_q \end{bmatrix}. \tag{3}$$

On substituting (2) into (1), we get

$$\begin{aligned} y(t) &= \sum_{i=0}^{k} \sum_{j=0}^{q} \gamma_j i^j x(t-i) + \varepsilon(t) \\ &= \sum_{j=0}^{q} \gamma_j z_j(t) + \varepsilon(t), \end{aligned} \tag{4}$$

where $z_j(t) = \sum_{i=0}^{k} i^j x(t-i)$. Given a set of T observations $(x_1, y_1), \ldots, (x_T, y_T)$, we can form the variables $z_{0t}, \ldots, z_{qt}$ for $t = 1, \ldots, T$. Then, by using the method of ordinary least squares to regress y_t on these variables, we can find estimates of the polynomial parameters $\gamma_0, \ldots, \gamma_q$. By substituting these estimates into the equation (2), we can find estimates of the lag coefficients $\beta_0, \ldots, \beta_k$.

If the equation of (1) were written in matrix form as $y = X\beta + \varepsilon$, then setting $\beta = \Pi\gamma$ would give

$$y = X\Pi\gamma + \varepsilon = Z\gamma + \varepsilon. \tag{5}$$

From the estimate $\hat{\gamma} = (Z'Z)^{-1}Z'y$ we should find $\hat{\beta} = \Pi\hat{\gamma}$.

In her original exposition, Almon employed an alternative form of the qth-degree polynomial $P(i)$. This takes account of the fact that, if $\tau_0, \ldots, \tau_q$ are any $q+1$ points in the domain of $P(i)$, then the polynomial is completely specified by knowing the values $p_0 = P(\tau_0), \ldots, p_q = P(\tau_q)$. Taking these values as our parameters, we can write the alternative representation as

$$P(\tau) = \sum_{j=0}^{q} p_j \delta_j(\tau), \tag{6}$$

wherein

$$\delta_j(\tau) = \frac{\prod_{\ell \neq j}^{q}(\tau - \tau_\ell)}{\prod_{l \neq j}^{q}(\tau_j - \tau_\ell)} \tag{7}$$

is a polynomial of degree q with the property that $\delta_j(\tau_j) = 1$ and $\delta_j(\tau_\ell) = 0$ for all $\ell \neq j$. The functions $\delta_j(\tau); j = 0, \ldots, q$ are called Lagrangean interpolation polynomials; and together they form a basis for the set of all polynomials of degrees less than or equal to q. A complete array of the $q+1$ basis function can be represented as follows:

$$\begin{aligned}
\delta_0(\tau) &= \frac{(\tau - \tau_1)(\tau - \tau_2)\cdots(\tau - \tau_q)}{(\tau_0 - \tau_1)(\tau_0 - \tau_2)\cdots(\tau_0 - \tau_q)}, \\
\delta_1(\tau) &= \frac{(\tau - \tau_0)(\tau - \tau_2)\cdots(\tau - \tau_q)}{(\tau_1 - \tau_0)(\tau_1 - \tau_2)\cdots(\tau_1 - \tau_q)}, \\
&\vdots \\
\delta_q(\tau) &= \frac{(\tau - \tau_0)(\tau - \tau_1)\cdots(\tau - \tau_{q-1})}{(\tau_q - \tau_0)(\tau_q - \tau_1)\cdots(\tau_q - \tau_{q-1})}.
\end{aligned} \tag{8}$$

If, instead of (2), we substitute into (1) the expression

$$\beta_i = P(i) = \sum_{j=0}^{q} p_j \delta_j(i), \tag{9}$$

then we obtain the equation

$$\begin{aligned}
y(t) &= \sum_{i=0}^{k}\sum_{j=0}^{q} p_j \delta_j(i) x(t-i) + \varepsilon(t) \\
&= \sum_{j=0}^{q} p_j z_j(t) + \varepsilon(t),
\end{aligned} \tag{10}$$

where $z_j(t) = \sum_{j=0}^{q} \delta_j(i)x(t-i)$. Given a set of T observations, we can estimate the lag coefficients in the same way a before.

First, we choose values $\tau_0, \ldots, \tau_q$. These should be a sequence of rising values in the vicinity of the interval $[0, k]$, which is the range of the index of tithe coefficients $\{\beta_i\}$. Then, having formed the associated Lagrangean interpolation polynomials, we can find the T values of the transformed variables $z_{0t}, \ldots, z_{qt}$. Next, the estimates of the polynomial parameters $p_0, \ldots, p_q$ are found by using the method of ordinary least squares to regress y_t on the transformed variables. Finally, the estimates of the lag coefficients $\beta_0, \ldots, \beta_q$ can be obtained from equation (9) by putting the estimates of the polynomial parameters in place of the unknown values.

The Lagrangean polynomials have the advantage that they enable us to tie down the sequence of coefficients $\beta_0, \ldots, \beta_k$ at either end by ensuring that $\beta_{-1} = P(-1) = 0$ and $\beta_{k+1} = P(k+1) = 0$. This is achieved by setting $\tau_0 = -1$ and $\tau_q = k+1$ and by specifying that $P(\tau_0) = 0$ and $P(\tau_q) = 0$. With these values in place, only $q-1$ ordinates $p_1, \ldots, p_{q-1}$ need to be estimated since equation (8) reduces to

$$(11) \qquad \beta_i = P(i) = \sum_{j=2}^{q} p_j \delta_j(\tau_j) \quad \text{where} \quad i = 0, \ldots, k.$$

The polynomial $P(i)$ can be expressed as a linear combination of other sets of basis polynomials. Although the choice of the Lagrangean interpolation polynomials as a basis renders the problem of incorporating certain types of *a priori* information particularly tractable, it may be less useful for incorporating other types of information.