LINEAR STOCHASTIC MODELS

Let $\{x_{\tau+1}, x_{\tau+2}, \ldots, x_{\tau+n}\}$ denote n consecutive elements from a stochastic process. If their joint distribution does not depend on τ, regardless of the size of n, then the process is strictly stationary. Any two segments of equal length will have the same distribution with

$$(1) \qquad E(x_t) = \mu < \infty \quad \text{for all} \quad t \quad \text{and} \quad C(x_{\tau+t}, x_{\tau+s}) = \gamma_{|t-s|}.$$

The condition on the covariances implies that the dispersion matrix of the vector $[x_1, x_2, \ldots, x_n]$ is a bisymmetric Laurent matrix of the form

$$(2) \qquad \Gamma = \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{n-1} \\ \gamma_1 & \gamma_0 & \gamma_1 & \cdots & \gamma_{n-2} \\ \gamma_2 & \gamma_1 & \gamma_0 & \cdots & \gamma_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_{n-1} & \gamma_{n-2} & \gamma_{n-3} & \cdots & \gamma_0 \end{bmatrix},$$

wherein the generic element in the (i, j)th position is $\gamma_{|i-j|} = C(x_i, x_j)$.

Moving-Average Processes

The qth-order moving average MA(q) process, is defined by

$$(3) \qquad y(t) = \mu_0\varepsilon(t) + \mu_1\varepsilon(t-1) + \cdots + \mu_q\varepsilon(t-q),$$

where $\varepsilon(t) = \{\varepsilon_t; t = 0, \pm1, \pm2, \ldots\}$ is a sequence of i.i.d. random variables with $E\{\varepsilon(t)\} = 0$ and $V(\varepsilon_t) = \sigma_\varepsilon^2$, defined on a doubly-infinite set of integers. We set can $\mu_0 = 1$.

The equation can also written as

$$y(t) = \mu(L)\varepsilon(t), \qquad \text{where} \qquad \mu(L) = \mu_0 + \mu_1 L + \cdots + \mu_q L^q$$

is a polynomial in the lag operator L, for which $L^j x(t) = x(t-j)$.

This process is stationary, since any two elements y_t and y_s are the same function of $[\varepsilon_t, \varepsilon_{t-1}, \ldots, \varepsilon_{t-q}]$ and $[\varepsilon_s, \varepsilon_{s-1}, \ldots, \varepsilon_{s-q}]$, which are identically distributed.

If the roots of the polynomial equation $\mu(z) = \mu_0 + \mu_1 z + \cdots + \mu_q z^q = 0$ lie outside the unit circle, then the process is invertible such that

$$\mu^{-1}(L)y(t) = \varepsilon(t),$$

which is an infinite-order autoregressive representation.

Example. Consider the first-order MA(1) moving-average process

$$(4) \qquad y(t) = \varepsilon(t) - \theta\varepsilon(t-1) = (1-\theta L)\varepsilon(t).$$

Provided that $|\theta| < 1$, this can be written in autoregressive form as

$$\varepsilon(t) = \frac{1}{(1-\theta L)}y(t) = \left\{y(t) + \theta y(t-1) + \theta^2 y(t-2) + \cdots\right\}.$$

Imagine that $|\theta| > 1$ instead. Then, to obtain a convergent series, we have to write

$$y(t+1) = \varepsilon(t+1) - \theta\varepsilon(t) = -\theta(1 - L^{-1}/\theta)\varepsilon(t),$$

where $L^{-1}\varepsilon(t) = \varepsilon(t+1)$. This gives

$$(7) \quad \varepsilon(t) = -\frac{\theta^{-1}}{(1-L^{-1}/\theta)}y(t+1) = -\theta^{-1}\left\{\frac{y(t+1)}{\theta} + \frac{y(t+2)}{\theta^2} + \cdots\right\}.$$

Normally, this would have no reasonable meaning.

The Autocovariances of a Moving-Average Process

Consider

$$
\begin{aligned}
\gamma_\tau &= E(y_t y_{t-\tau}) \\
&= E\Big\{ \sum_i \mu_i \varepsilon_{t-i} \sum_j \mu_j \varepsilon_{t-\tau-j} \Big\} \\
&= \sum_i \sum_j \mu_i \mu_j E(\varepsilon_{t-i} \varepsilon_{t-\tau-j}).
\end{aligned} \tag{8}
$$

Since $\varepsilon(t)$ is a sequence of independently and identically distributed random variables with zero expectations, it follows that

$$
E(\varepsilon_{t-i}\varepsilon_{t-\tau-j}) = \begin{cases} 0, & \text{if } i \neq \tau + j; \\ \sigma_\varepsilon^2, & \text{if } i = \tau + j. \end{cases} \tag{9}
$$

Therefore

$$
\gamma_\tau = \sigma_\varepsilon^2 \sum_j \mu_j \mu_{j+\tau}. \tag{10}
$$

Now let $\tau = 0, 1, \ldots, q$. This gives

$$(11) \qquad \begin{aligned} \gamma_0 &= \sigma_\varepsilon^2(\mu_0^2 + \mu_1^2 + \cdots + \mu_q^2), \\ \gamma_1 &= \sigma_\varepsilon^2(\mu_0\mu_1 + \mu_1\mu_2 + \cdots + \mu_{q-1}\mu_q), \\ &\vdots \\ \gamma_q &= \sigma_\varepsilon^2\mu_0\mu_q. \end{aligned}$$

Also, $\gamma_\tau = 0$ for all $\tau > q$.

The first-order moving-average process $y(t) = \varepsilon(t) - \theta\varepsilon(t-1)$ has the following autocovariances:

$$(12) \qquad \begin{aligned} \gamma_0 &= \sigma_\varepsilon^2(1 + \theta^2), \\ \gamma_1 &= -\sigma_\varepsilon^2\theta, \\ \gamma_\tau &= 0 \quad \text{if} \quad \tau > 1. \end{aligned}$$

For a vector $y = [y_0, y_2, \ldots, y_{T-1}]'$ of T consecutive elements from a first-order moving-average process, the dispersion matrix is

$$D(y) = \sigma_\varepsilon^2 \begin{bmatrix} 1+\theta^2 & -\theta & 0 & \ldots & 0 \\ -\theta & 1+\theta^2 & -\theta & \ldots & 0 \\ 0 & -\theta & 1+\theta^2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1+\theta^2 \end{bmatrix}. \tag{13}$$

In general, the dispersion matrix of a qth-order moving-average process has q subdiagonal and q supradiagonal bands of nonzero elements and zero elements elsewhere.

The empirical autocovariance of lag $\tau \leq T-1$ is

$$c_\tau = \frac{1}{T}\sum_{t=0}^{T-\tau}(y_t - \bar{y})(y_{t+\tau} - \bar{y}) \qquad \text{with} \qquad \bar{y} = \frac{1}{T}\sum_{t=0}^{T-1} y_t.$$

Notice that $c_{T-1} = T^{-1}y_0 y_{T-1}$ comprises only the first and the last element of the sample.

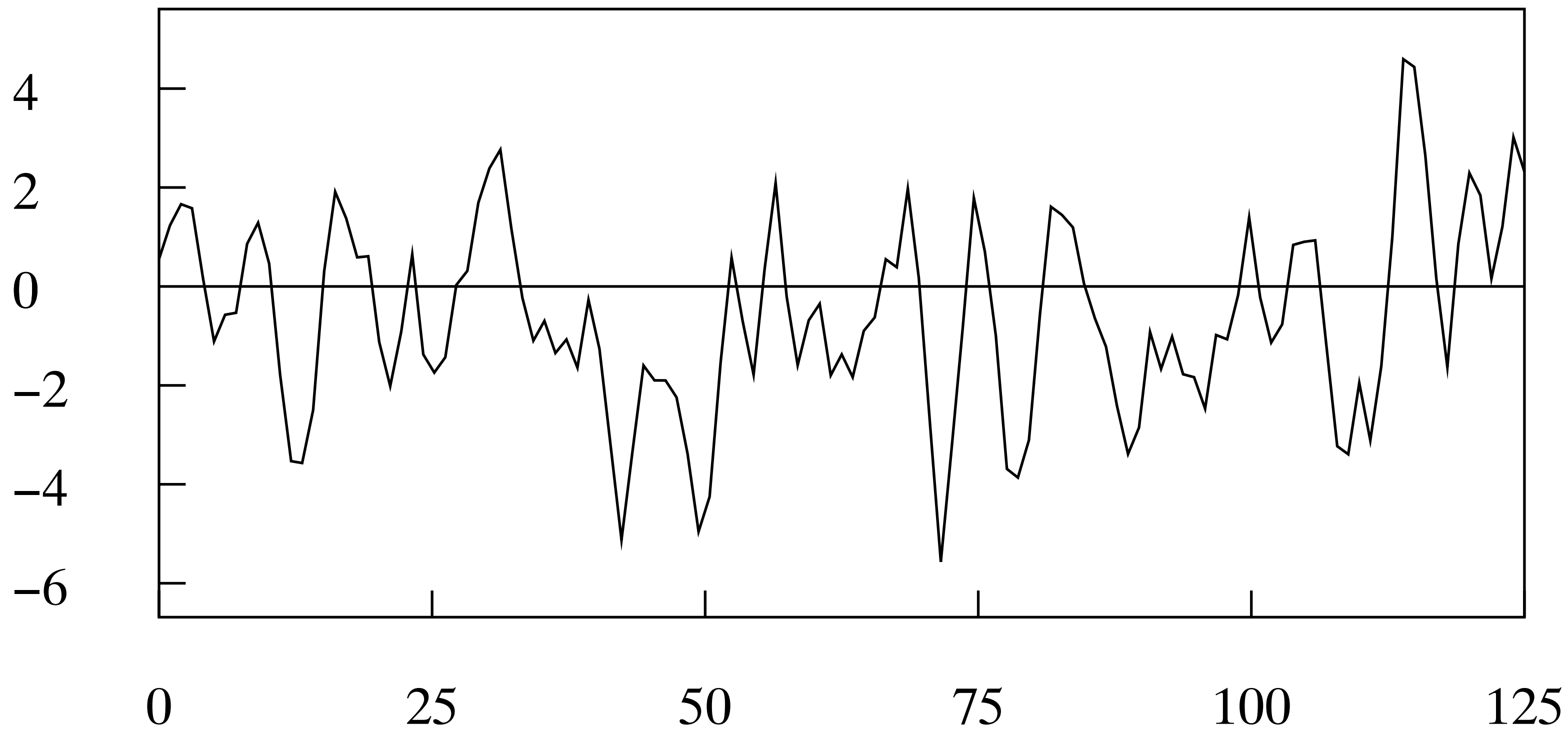

Figure 1. The graph of 125 observations on a simulated series generated by an MA(2) process $y(t) = (1 + 1.25L + 0.80L^2)\varepsilon(t)$.

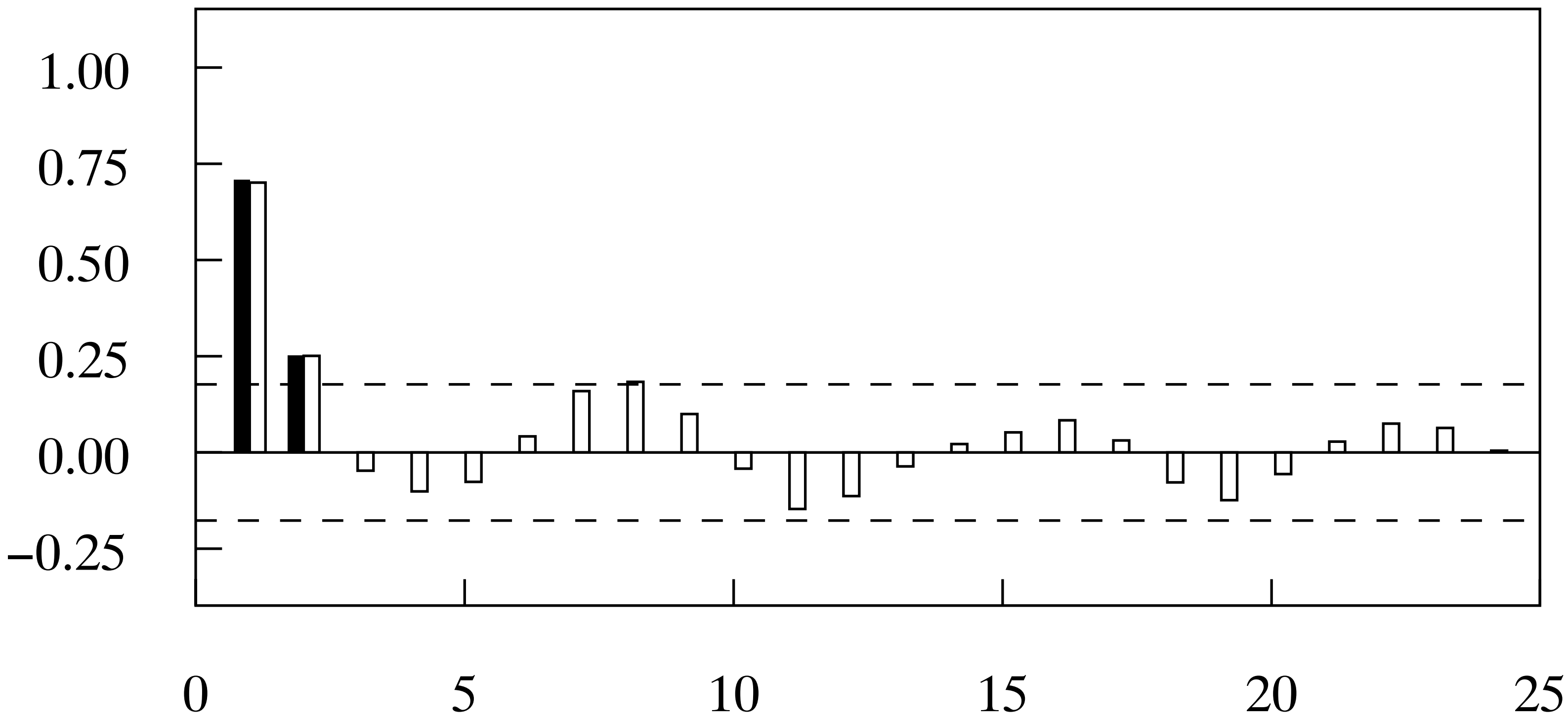

Figure 2. The theoretical autocorrelations of the MA(2) process $y(t) = (1 + 1.25L + 0.80L^2)\varepsilon(t)$ (the solid bars) together with their empirical counterparts, calculated from a simulated series of 125 values.

Autoregressive Processes

The pth-order autoregressive AR(p) process, is defined by

$$(17) \qquad \alpha_0 y(t) + \alpha_1 y(t-1) + \cdots + \alpha_p y(t-p) = \varepsilon(t).$$

Setting $\alpha_0 = 1$ identifies $y(t)$ as the output. This can be written as

$$\alpha(L)y(t) = \varepsilon(t), \qquad \text{where} \qquad \alpha(L) = \alpha_0 + \alpha_1 L + \cdots + \alpha_p L^p.$$

For the process to be stationary, the roots of the equation $\alpha(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_p z^p = 0$ must lie outside the unit circle.

This condition enables us to write the autoregressive process as an infinite-order moving-average process in the form of

$$y(t) = \alpha^{-1}(L)\varepsilon(t).$$

Example. Consider the AR(1) process defined by

$$
\begin{aligned}
\varepsilon(t) &= y(t) - \phi y(t-1) \\
&= (1 - \phi L)y(t).
\end{aligned}
\tag{18}
$$

Provided that the process is stationary with $|\phi| < 1$, it can be represented in moving-average form as

$$
y(t) = \frac{1}{1 - \phi L}\varepsilon(t) = \big\{\varepsilon(t) + \phi\varepsilon(t-1) + \phi^2\varepsilon(t-2) + \cdots\big\}.
\tag{19}
$$

The autocovariances of the AR(1) process can be found in the manner of an MA process. Thus

$$
\begin{aligned}
\gamma_\tau &= E(y_t y_{t-\tau}) \\
&= E\Big\{\sum_i \phi^i \varepsilon_{t-i} \sum_j \phi^j \varepsilon_{t-\tau-j}\Big\} \\
&= \sum_i \sum_j \phi^i \phi^j E(\varepsilon_{t-i}\varepsilon_{t-\tau-j});
\end{aligned}
\tag{20}
$$

Since

$$E(\varepsilon_{t-i}\varepsilon_{t-\tau-j}) = \begin{cases} 0, & \text{if } i \neq \tau + j; \\ \sigma_\varepsilon^2, & \text{if } i = \tau + j, \end{cases} \tag{9}$$

it follows that

$$\gamma_\tau = \sigma_\varepsilon^2 \sum_j \phi^j \phi^{j+\tau} = \frac{\sigma_\varepsilon^2 \phi^\tau}{1 - \phi^2}. \tag{21}$$

For a vector $y = [y_0, y_2, \ldots, y_{T-1}]'$ of T consecutive elements from a first-order autoregressive process, the dispersion matrix has the form

$$D(y) = \frac{\sigma_\varepsilon^2}{1 - \phi^2} \begin{bmatrix} 1 & \phi & \phi^2 & \ldots & \phi^{T-1} \\ \phi & 1 & \phi & \ldots & \phi^{T-2} \\ \phi^2 & \phi & 1 & \ldots & \phi^{T-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi^{T-1} & \phi^{T-2} & \phi^{T-3} & \ldots & 1 \end{bmatrix}. \tag{22}$$

The Autocovariances of an Autoregressive Process

Multiplying $\sum_i \alpha_i y_{t-i} = \varepsilon_t$ by $y_{t-\tau}$ and taking expectations gives

(24) $$\sum_i \alpha_i E(y_{t-i}y_{t-\tau}) = E(\varepsilon_t y_{t-\tau}).$$

Taking account of the normalisation $\alpha_0 = 1$, we find that

(25) $$E(\varepsilon_t y_{t-\tau}) = \begin{cases} \sigma_\varepsilon^2, & \text{if } \tau = 0; \\ 0, & \text{if } \tau > 0. \end{cases}$$

Therefore, on setting $E(y_{t-i}y_{t-\tau}) = \gamma_{\tau-i}$, equation (24) gives

(26) $$\sum_i \alpha_i \gamma_{\tau-i} = \begin{cases} \sigma_\varepsilon^2, & \text{if } \tau = 0; \\ 0, & \text{if } \tau > 0. \end{cases}$$

The second equation enables us to generate the sequence $\{\gamma_p, \gamma_{p+1}, \ldots\}$ given p starting values $\gamma_0, \gamma_1, \ldots, \gamma_{p-1}$.

According to (26), there is

$$\alpha_0\gamma_\tau + \alpha_1\gamma_{\tau-1} + \cdots + \alpha_2\gamma_{\tau-p} = 0 \quad \text{for} \quad \tau > 0$$

Thus, given $\gamma_{\tau-1}, \gamma_{\tau-2}, \ldots, \gamma_{\tau-p}$ for $\tau \geq p$, we can find

$$\gamma_\tau = -\alpha_1\gamma_{\tau-1} - \alpha_2\gamma_{\tau-2} - \cdots - \alpha_p\gamma_{\tau-p}.$$

By letting $\tau = 0, 1, \ldots, p$ in (26), we generate a set of $p+1$ equations, which can be arrayed in matrix form as follows:

$$(27) \qquad \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_p \\ \gamma_1 & \gamma_0 & \gamma_1 & \cdots & \gamma_{p-1} \\ \gamma_2 & \gamma_1 & \gamma_0 & \cdots & \gamma_{p-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_p & \gamma_{p-1} & \gamma_{p-2} & \cdots & \gamma_0 \end{bmatrix} \begin{bmatrix} 1 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_p \end{bmatrix} = \begin{bmatrix} \sigma_\varepsilon^2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

These the Yule–Walker equations, which can be used for generating the values $\gamma_0, \gamma_1, \ldots, \gamma_p$ from the values $\alpha_1, \ldots, \alpha_p, \sigma_\varepsilon^2$ or vice versa.

Example. For an example of the two uses of the Yule–Walker equations, consider the AR(2) process. In this case,

$$
(28)\qquad
\begin{aligned}
&\begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 \\ \gamma_1 & \gamma_0 & \gamma_1 \\ \gamma_2 & \gamma_1 & \gamma_0 \end{bmatrix}
\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix}
=
\begin{bmatrix} \alpha_2 & \alpha_1 & \alpha_0 & 0 & 0 \\ 0 & \alpha_2 & \alpha_1 & \alpha_0 & 0 \\ 0 & 0 & \alpha_2 & \alpha_1 & \alpha_0 \end{bmatrix}
\begin{bmatrix} \gamma_2 \\ \gamma_1 \\ \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{bmatrix} \\
&=
\begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 \\ \alpha_1 & \alpha_0+\alpha_2 & 0 \\ \alpha_2 & \alpha_1 & \alpha_0 \end{bmatrix}
\begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{bmatrix}
=
\begin{bmatrix} \sigma_\varepsilon^2 \\ 0 \\ 0 \end{bmatrix}.
\end{aligned}
$$

Given $\alpha_0 = 1$ and the values for $\gamma_0, \gamma_1, \gamma_2$, we can find σ_ε^2 and α_1, α_2. Conversely, given $\alpha_0, \alpha_1, \alpha_2$ and σ_ε^2, we can find $\gamma_0, \gamma_1, \gamma_2$.

Notice how the matrix following the first equality is folded across the axis which divides it vertically to give the matrix which follows the second equality.

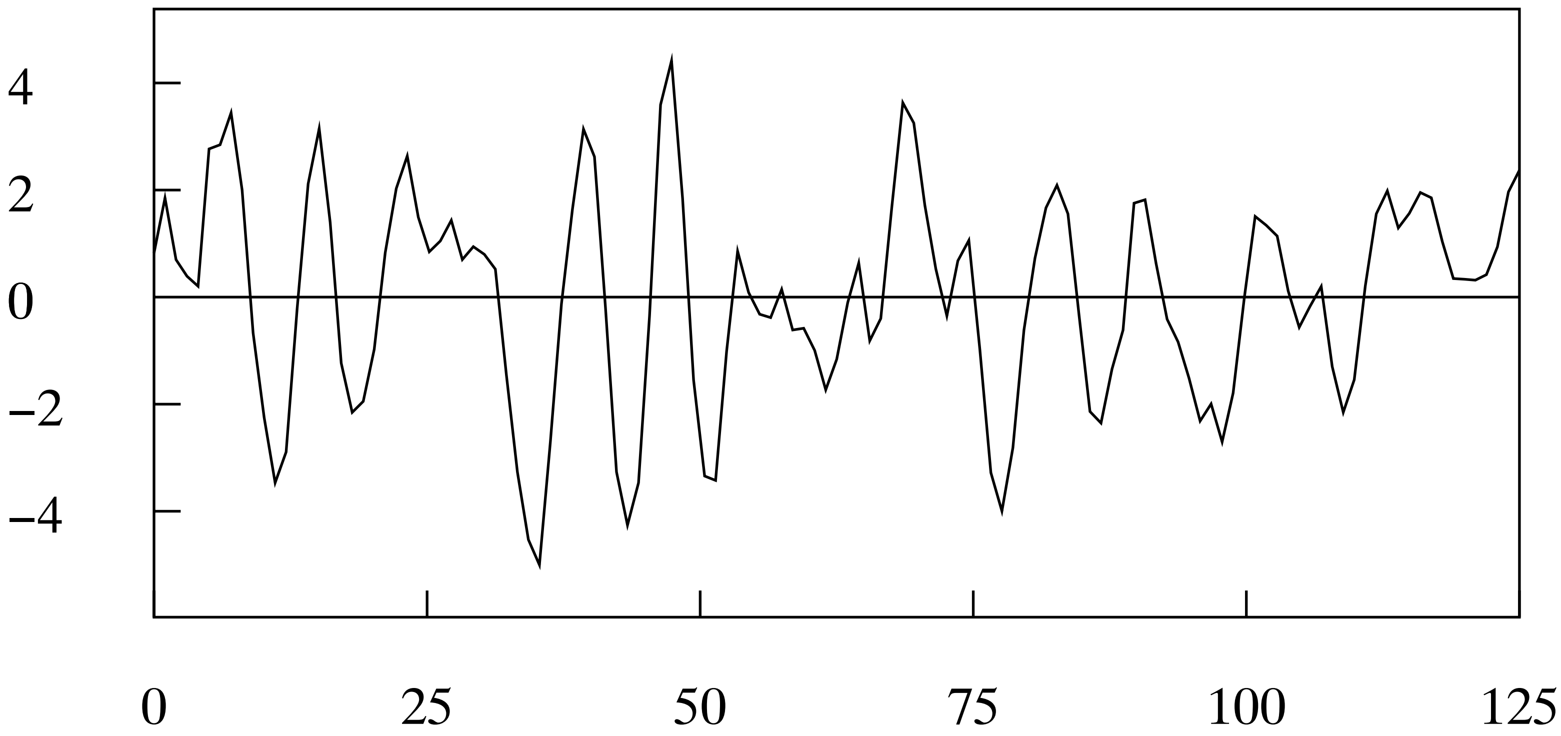

Figure 3. The graph of 125 observations on a simulated series generated by an AR(2) process $(1 - 0.273L + 0.81L^2)y(t) = \varepsilon(t)$.

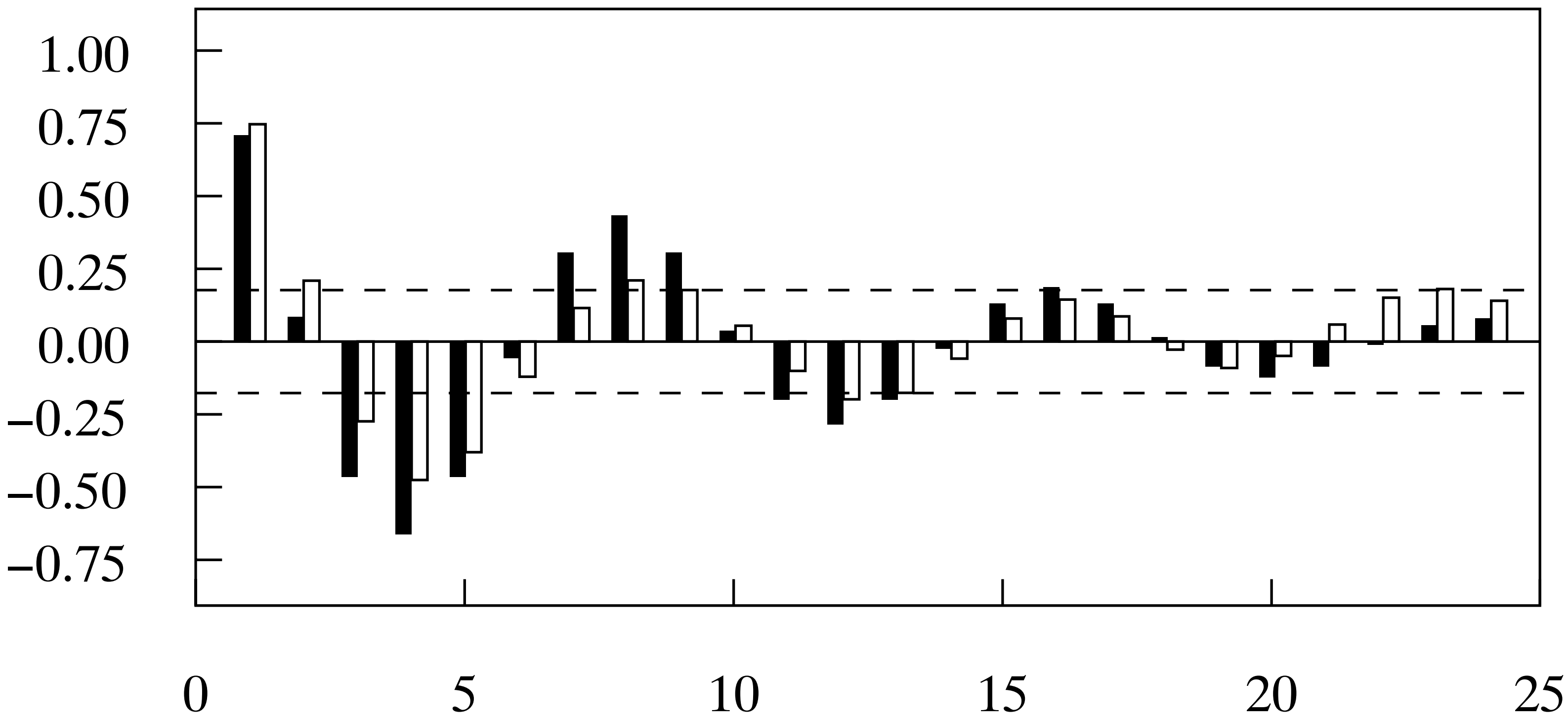

Figure 4. The theoretical autocorrelations and of the AR(2) process $(1 - 0.273L + 0.81L^2)y(t) = \varepsilon(t)$ (the solid bars) together with their empirical counterparts, calculated from a simulated series of 125 values.

The Partial Autocorrelation Function

Let $\alpha_{r(r)}$ be the coefficient associated with $y(t-r)$ in an autoregressive process of order r whose parameters correspond to the autocovariances $\gamma_0, \gamma_1, \ldots, \gamma_r$. Then the sequence $\{\alpha_{r(r)}; r = 1, 2, \ldots\}$, of which the index corresponds to models of increasing orders, constitutes the partial autocorrelation function.

In effect, $\alpha_{r(r)}$ indicates the role in explaining the variance of $y(t)$ which is due to $y(t-r)$ when $y(t-1), \ldots, y(t-r+1)$ are also taken into account.

The sample partial autocorrelation p_τ at lag τ is the correlation between the two sets of residuals obtained from regressing the elements y_t and $y_{t-\tau}$ on the set of intervening values $y_{t-1}, y_{t-2}, \ldots, y_{t-\tau+1}$. The partial autocorrelation measures the dependence between y_t and $y_{t-\tau}$ after the effect of the intervening values has been removed.

The theoretical partial autocorrelations function of a AR(p) process is zero-valued for all $\tau > p$. Likewise, all elements of the sample partial autocorrelation function are expected to be close to zero for lags greater than p

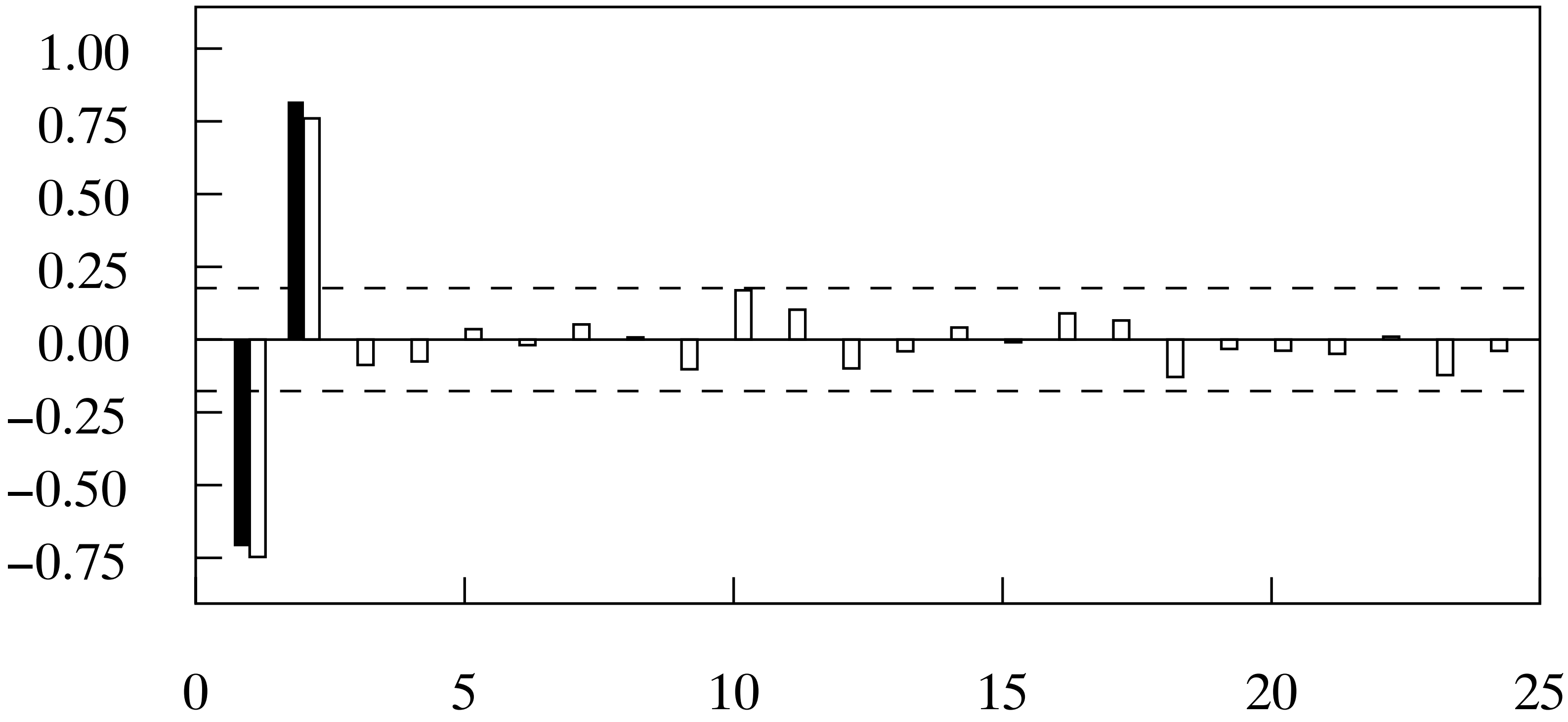

Figure 5. The theoretical partial autocorrelations of the AR(2) process $(1 - 0.273L + 0.81L^2)y(t) = \varepsilon(t)$ together with their empirical counterparts, calculated from a simulated series of 125 values.

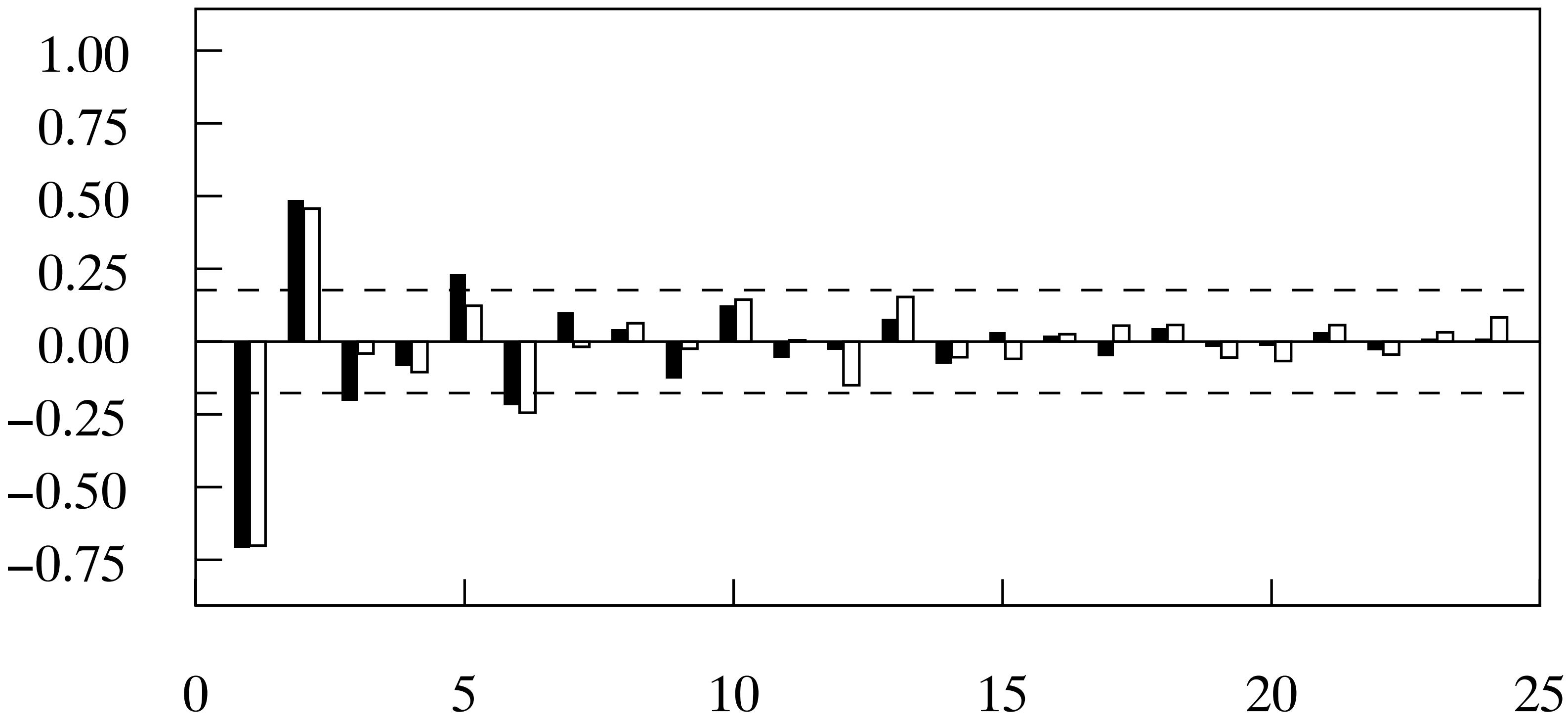

Figure 6. The theoretical partial autocorrelations of the MA(2) process $y(t) = (1 + 1.25L + 0.80L^2)\varepsilon(t)$ together with their empirical counterparts, calculated from a simulated series of 125 values.

Autoregressive Moving-Average Processes

The autoregressive moving-average ARMA(p, q) process of orders p and q is defined by

$$\begin{aligned}(36)\qquad &\alpha_0 y(t) + \alpha_1 y(t-1) + \cdots + \alpha_p y(t-p)\\ &\quad = \mu_0 \varepsilon(t) + \mu_1 \varepsilon(t-1) + \cdots + \mu_q \varepsilon(t-q).\end{aligned}$$

The equation is normalised by setting $\alpha_0 = 1$ and $\mu_0 = 1$. The equation can be denoted by

$$\alpha(L)y(t) = \mu(L)\varepsilon(t).$$

Provided that the roots of the equation $\alpha(z) = 0$ lie outside the unit circle, the process can be described as an infinite-order MA process:

$$y(t) = \alpha^{-1}(L)\mu(L)\varepsilon(t).$$

Conversely, provided the roots of the equation $\mu(z) = 0$ lie outside the unit circle, the process can be described as an infinite-order AR process:

$$\mu^{-1}(L)\alpha(L)y(t) = \varepsilon(t).$$

The Autocovariances of an ARMA Process

Multiplying $\sum_i \alpha_i y_{t-i} = \sum_i \mu_i \varepsilon_{t-i}$ by $y_{t-\tau}$ and taking expectations gives

$$\sum_i \alpha_i \gamma_{\tau-i} = \sum_i \mu_i \delta_{i-\tau}, \tag{38}$$

where $\gamma_{\tau-i} = E(y_{t-\tau} y_{t-i})$ and $\delta_{i-\tau} = E(y_{t-\tau}\varepsilon_{t-i})$. Since ε_{t-i} is uncorrelated with $y_{t-\tau}$ whenever it is subsequent to the latter, it follows that $\delta_{i-\tau} = 0$ if $\tau > i$.

Since the index i in the RHS of the equation (38) runs from 0 to q, it follows that

$$\sum_i \alpha_i \gamma_{i-\tau} = 0 \quad \text{if} \quad \tau > q. \tag{39}$$

Given the $q+1$ values $\delta_0, \delta_1, \ldots, \delta_q$, and p initial values $\gamma_0, \gamma_1, \ldots, \gamma_{p-1}$ for the autocovariances, the equation (38) can be solved recursively to obtain the subsequent values $\{\gamma_p, \gamma_{p+1}, \ldots\}$.

To find the requisite values $\delta_0, \delta_1, \ldots, \delta_q$, consider multiplying the equation $\sum_i \alpha_i y_{t-i} = \sum_i \mu_i \varepsilon_{t-i}$ by $\varepsilon_{t-\tau}$ and taking expectations. This gives

$$\sum_i \alpha_i \delta_{\tau-i} = \mu_\tau \sigma_\varepsilon^2, \tag{40}$$

where $\delta_{\tau-i} = E(y_{t-i}\varepsilon_{t-\tau})$. The equation may be rewritten as

$$\delta_\tau = \frac{1}{\alpha_0}\Big(\mu_\tau \sigma_\varepsilon^2 - \sum_{i=1} \delta_{\tau-i}\Big), \tag{41}$$

and, by setting $\tau = 0, 1, \ldots, q$, we can generate recursively the required values $\delta_0, \delta_1, \ldots, \delta_q$.

Example. Consider the ARMA(2, 2) model, which gives the equation

$$\alpha_0 y_t + \alpha_1 y_{t-1} + \alpha_2 y_{t-2} = \mu_0\varepsilon_t + \mu_1\varepsilon_{t-1} + \mu_2\varepsilon_{t-2}. \tag{42}$$

Multiplying by y_t, y_{t-1} and y_{t-2} and taking expectations gives

$$\begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 \\ \gamma_1 & \gamma_0 & \gamma_1 \\ \gamma_2 & \gamma_1 & \gamma_0 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \delta_0 & \delta_1 & \delta_2 \\ 0 & \delta_0 & \delta_1 \\ 0 & 0 & \delta_0 \end{bmatrix} \begin{bmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \end{bmatrix}. \tag{43}$$

Multiplying by ε_t, ε_{t-1} and ε_{t-2} and taking expectations gives

$$\begin{bmatrix} \delta_0 & 0 & 0 \\ \delta_1 & \delta_0 & 0 \\ \delta_2 & \delta_1 & \delta_0 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \sigma_\varepsilon^2 & 0 & 0 \\ 0 & \sigma_\varepsilon^2 & 0 \\ 0 & 0 & \sigma_\varepsilon^2 \end{bmatrix} \begin{bmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \end{bmatrix}. \tag{44}$$

When the latter equations are written as

$$\begin{bmatrix} \alpha_0 & 0 & 0 \\ \alpha_1 & \alpha_0 & 0 \\ \alpha_2 & \alpha_1 & \alpha_0 \end{bmatrix} \begin{bmatrix} \delta_0 \\ \delta_1 \\ \delta_2 \end{bmatrix} = \sigma_\varepsilon^2 \begin{bmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \end{bmatrix}, \tag{45}$$

they can be solved recursively for δ_0, δ_1 and δ_2 on the assumption that that the values of α_0, α_1, α_2 and σ_ε^2 are known. Notice that, when we adopt the normalisation $\alpha_0 = \mu_0 = 1$, we get $\delta_0 = \sigma_\varepsilon^2$. When the equations (43) are rewritten as

$$(46) \qquad \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 \\ \alpha_1 & \alpha_0 + \alpha_2 & 0 \\ \alpha_2 & \alpha_1 & \alpha_0 \end{bmatrix} \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} \mu_0 & \mu_1 & \mu_2 \\ \mu_1 & \mu_2 & 0 \\ \mu_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_0 \\ \delta_1 \\ \delta_2 \end{bmatrix},$$

they can be solved for γ_0, γ_1 and γ_2. Thus the starting values are obtained, which enable the equation

$$(47) \qquad \alpha_0\gamma_\tau + \alpha_1\gamma_{\tau-1} + \alpha_2\gamma_{\tau-2} = 0; \quad \tau > 2$$

to be solved recursively to generate the succeeding values $\{\gamma_3, \gamma_4, \ldots\}$ of the autocovariances.

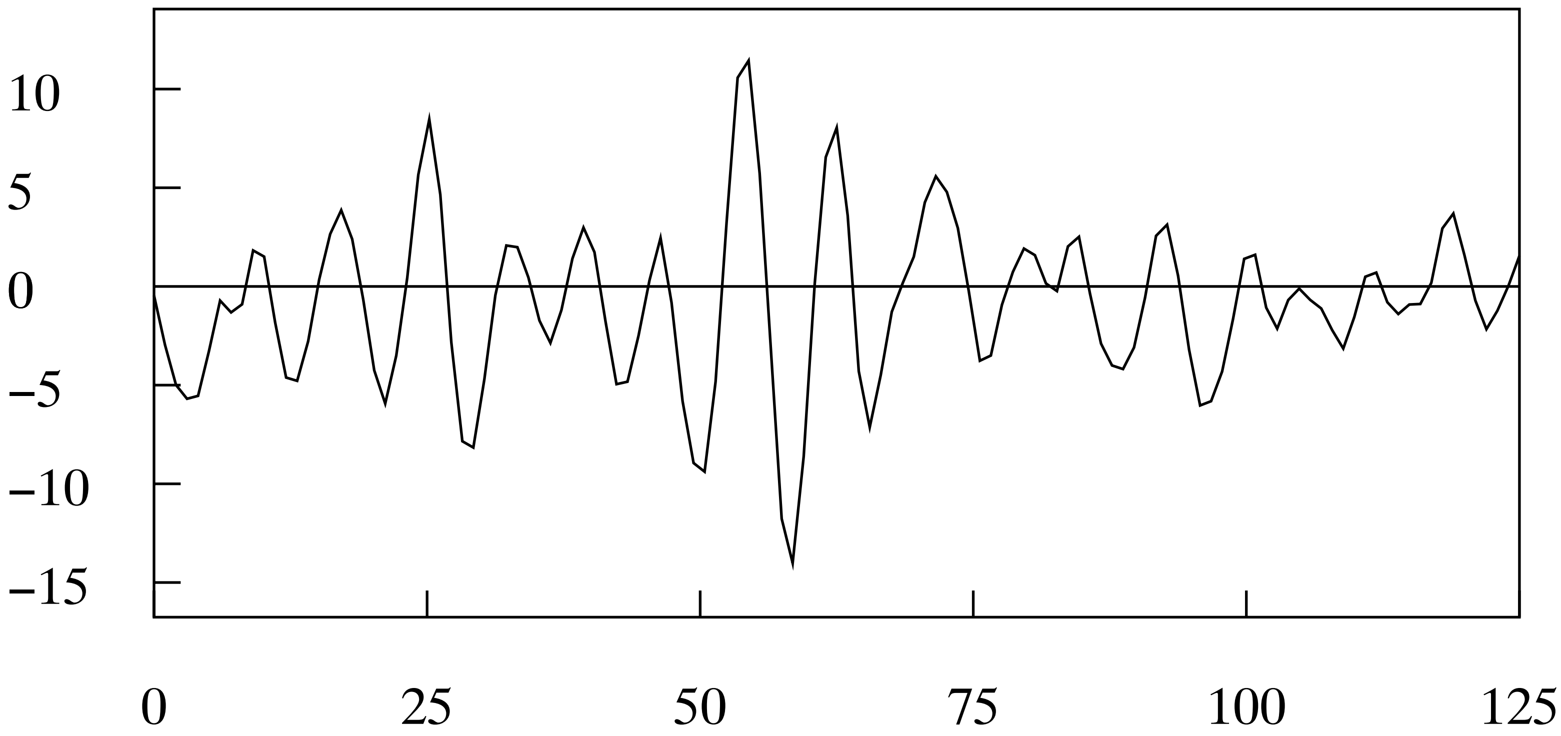

Figure 7. The graph of 125 observations on a simulated series generated by an ARMA(2, 1) process $(1 - 0.273L + 0.81L^2)y(t) = (1 + 0.9L)\varepsilon(t)$.

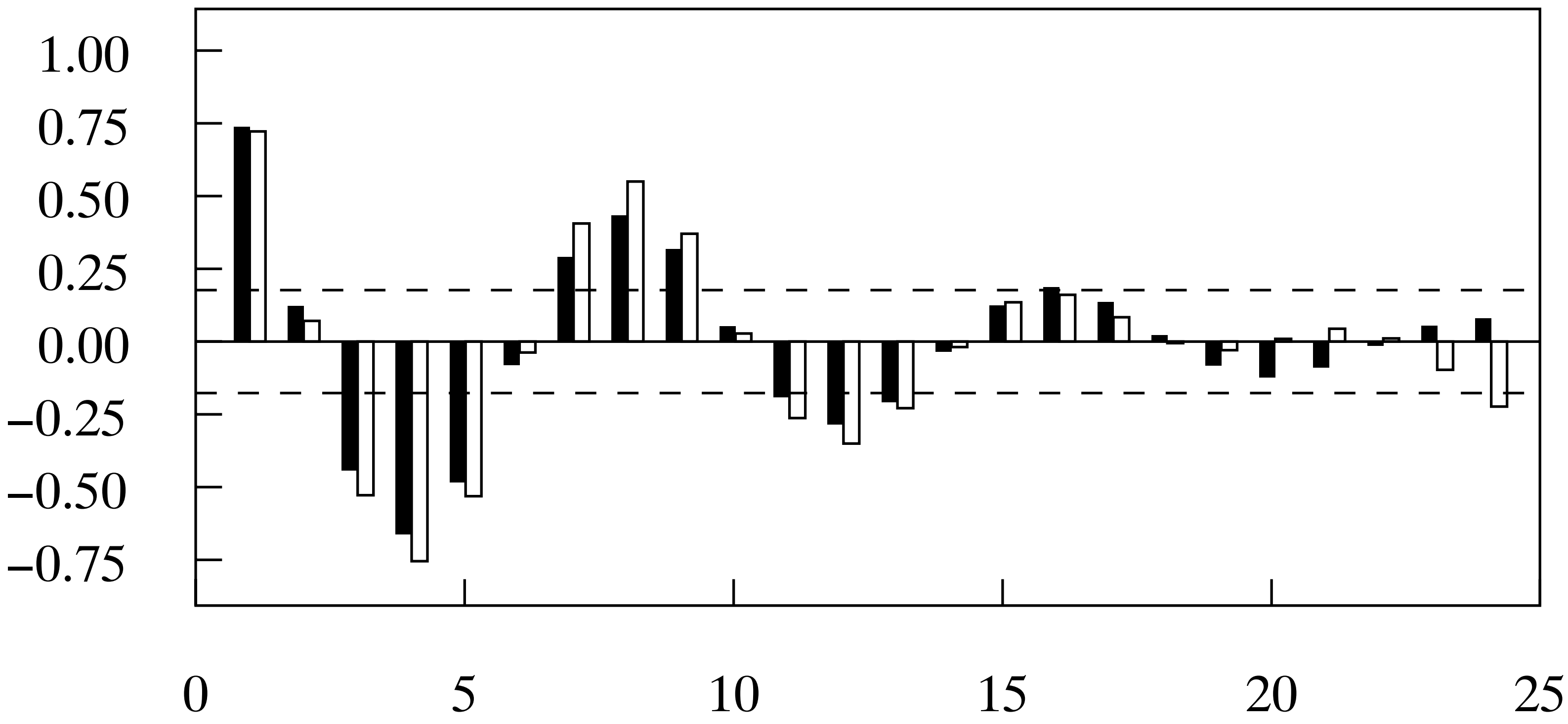

Figure 8. The theoretical autocorrelations and of the ARMA(2, 1) process $(1 - 0.273L + 0.81L^2)y(t) = (1 + 0.9L)\varepsilon(t)$ together with their empirical counterparts, calculated from a simulated series of 125 values.

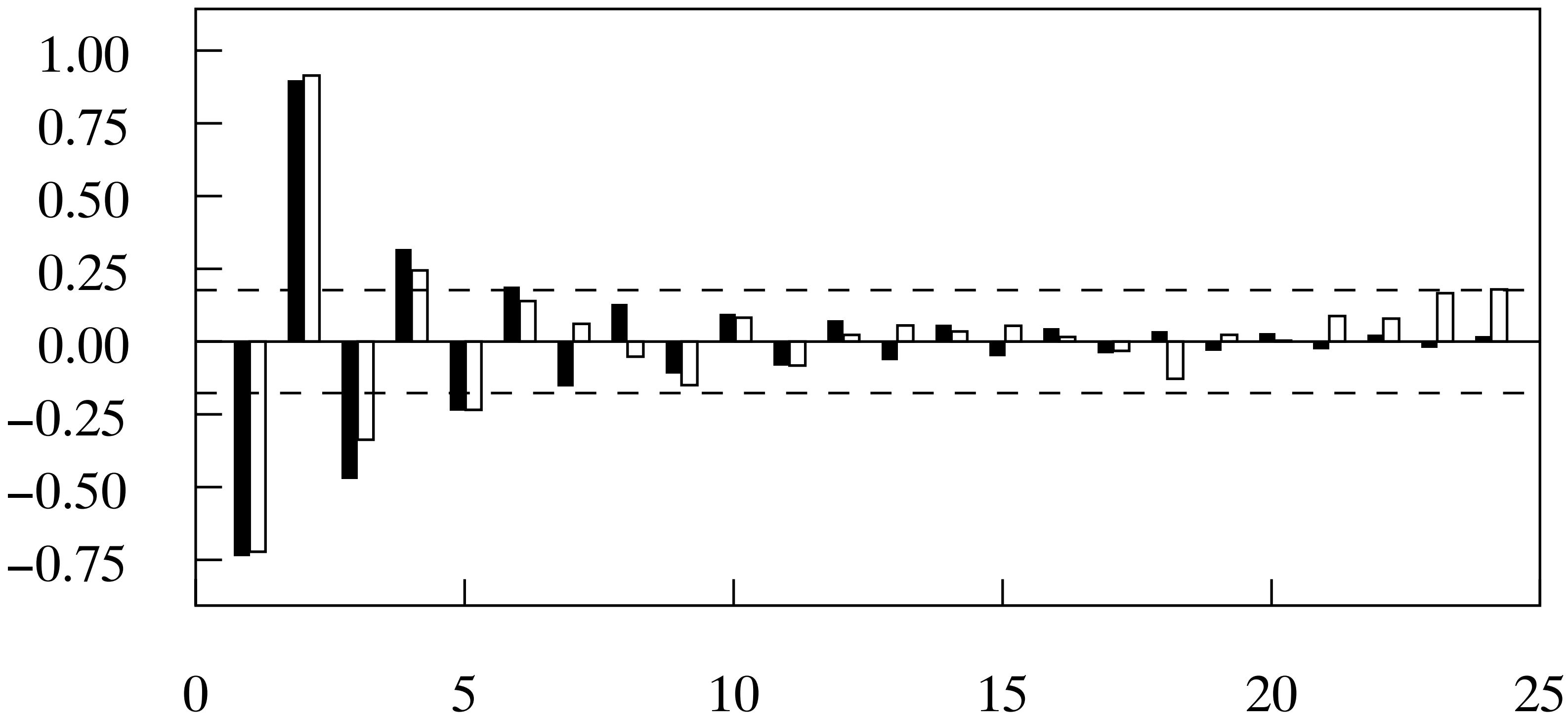

Figure 9. The theoretical partial autocorrelations of the ARMA(2, 1) process $(1 - 0.273L + 0.81L^2)y(t) = (1 + 0.9L)\varepsilon(t)$ together with their empirical counterparts, calculated from a simulated series of 125 values.