## APPENDIX 8

# Transfer Functions

In the realms of statistical time series analysis, of signal processing and of control engineering, a transfer function is a mathematical relationship between the numerical input to a dynamic system and the resulting output.

The theory concerning the transfer functions of linear time-invariant systems, which is the subject of the present paper, has been available for many years. It was developed originally in connection with electrical and mechanical systems described in continuous time. The basic theory can be attributed in large measure to Oliver Heaviside (1850–1925)

With the advent of digital signal processing, the emphasis has shifted to discrete-time representations. These are also appropriate to problems in statistical time series analysis, where it is commonly supposed that the data are in the form of sequences of values sampled at regular intervals.

The Nyquist–Shannon sampling theorem establishes the connection between the continuous-time and the discrete-time representations of a linear system; and it indicates that, if the sampling is at a sufficient rate, then the discrete representation will contain all of the relevant information on the system.

In the discrete case, a univariate and strictly causal transfer function mapping from the input sequence $\{x_t\}$ to the output sequence $\{y_t;\}$ can be represented by the equation

$$\sum_{j=0}^{p} a_j y_{t-j} = \sum_{j=0}^{q} b_j x_{t-j}, \quad \text{with} \quad a_0 = 1. \tag{1}$$

Here, the condition that $a_0 = 1$ serves to identify $y_t$ as the current output and the elements $y_{t-1}, \ldots, y_{t-p}$ as feedback or as lagged dependent variables. The sum of the input variables on RHS of the equation, weighted by their coefficients, is described as a distributed lag scheme. The condition of causality implies that $x_{t+j}$ and $y_{t+j}$, which are ahead of time $t$, are excluded from the equation.

Consider $T$ realisations of the equation (1) indexed by $t = 0, \ldots, T-1$. By associating each equation with the corresponding power $z^t$ of an indeterminate algebraic symbol $z$ and by adding them together, a polynomial equation is derived that can be denoted by

$$(2) \qquad a(z)y(z) = b(z)x(z) \quad \text{or, equivalently,} \quad y(z) = \frac{b(z)}{a(z)}x(z).$$

Here,

$$(3) \qquad \begin{aligned} y(z) &= y_{-p}z^{-p} + \cdots + y_0 + y_1 z + \cdots + y_{T-1}z^{T-1}, \\ x(z) &= x_{-q}z^{-q} + \cdots + x_0 + x_1 z + \cdots + x_{T-1}z^{T-1}, \\ a(z) &= 1 + a_1 z + \cdots + a_p z^p \quad \text{and} \\ b(z) &= b_0 + b_1 z + \cdots + b_q z^q. \end{aligned}$$

are described as the $z$-transforms of the corresponding sequences. The advantage of this representation is that it allows us to deploy the algebra of polynomials in analysing the dynamic system.

Observe that the equation comprises the pre-sample element $y_{-p}, \ldots, y_{-1}$ and $x_{-q}, \ldots, y_{-1}$ of the input and output sequences, which provide initial conditions for the system. However, on the supposition that the transfer function is stable in the sense that a bounded input will lead to a bounded output—described as the BIBO condition—it is permissible to extend the two sequences backward in time indefinitely and forwards in time. Then, $y(z)$ and $x(z)$ become infinite series; and the matter of the initial conditions can be ignored.

### The Impulse Response

The properties of the transfer function can be characterised by its effect on certain elementary reference signals. The simplest of these is the impulse sequence, which is an indefinite sequence defined by

$$(4) \qquad \delta_t = \begin{cases} 1, & \text{if } t = 0; \\ 0, & \text{if } t \neq 0. \end{cases}$$

The corresponding $z$-transform is $\delta(z) = 1$. The output engendered by the impulse is described as the impulse response function. For an ordinary causal transfer function, which responds only to present and previous values of the input and output sequences, it is sufficient to ignore the zero-valued response at times $t < 0$.

On substituting $\delta(z) = 1$ into equation (2), it can be seen that calculating the impulse response is a matter of finding coefficients of the series expansion of the rational function $b(z)/a(z)$. When $a(z) = 1$, it is manifest that the impulse response is just the sequence of the coefficients of $b(z)$; and it is said that this is a finite impulse response (FIR) transfer function. When $a(z) \neq 1$, the impulse response is liable to continue indefinitely, albeit that, in a stable system, it will converge to zero. Then, it is said that there is an infinite impulse response (IIR) transfer function.

In the case where $a(z) \neq 1$, we may consider the equation $b(z) = a(z)y(z)$, where $y(z) = \{y_0 + y_1 z + \cdots\}$. By forming the product on the RHS and equating the coefficients of the powers of $z$ with the coefficients of the same powers on the LHS, a sequence of equations is generated that can be solved recursively to find the response. An example is provided by the case where $p = 3$ and $q = 2$. Then

$$b_0 + b_1 z = \{1 + a_1 z + a_2 z^2\}\{y_0 + y_1 z + y_2 z^2 + \cdots\}. \tag{4}$$

By performing the multiplication on the RHS, and by equating the coefficients of the same powers of $z$ on the two sides of the equation, it is found that

$$\begin{aligned} b_0 &= y_0, & y_0 &= b_0, \\ b_1 &= y_1 + a_1 y_0, & y_1 &= b_1 - a_1 y_0, \\ 0 &= y_2 + a_1 y_1 + a_2 y_0, & y_2 &= -a_1 y_1 - a_2 y_0, \\ &\vdots & &\vdots \\ 0 &= y_n + a_1 y_{n-1} + a_2 y_{n-2}, & y_n &= -a_1 y_{n-1} - a_2 y_{n-2}. \end{aligned} \tag{5}$$

The resulting sequence is just the recursive solution of the homogenous difference equation $y_t + a_1 y_{t-1} + a_2 y_{t-2} = 0$, subject to the initial conditions $y_0 = b_0$ and $y_1 = b_1 - a_1 y_0$.

It is worth remarking that the transfer function is characterised in full by its response to an impulse. One is reminded that the harmonics of a bell are fully revealed when it is stuck by a single blow of the clapper, which constitutes an impulse.

Another reference sign that is commonly considered is that unit step defined by

$$s_t = \begin{cases} 1, & \text{if } t \geq 0; \\ 0, & \text{if } t < 0. \end{cases} \tag{6}$$

It is evident that the resulting step response is just the sequence of the partial sums of the impulse response.

## Stability

The stability of a rational transfer function $b(z)/a(z)$ can be investigated using its partial fraction decomposition, which gives rise to a sum of simpler transfer functions that are readily amenable to analysis.

If the degree of the denominator of $b(z)/a(z)$ exceeds that of the numerator, then long division can be used to obtain a quotient polynomial and a remainder that is a proper rational function. The quotient polynomial will correspond a stable transfer function and remainder will be amenable to a decomposition.

Assume that $b(z)/b(z)$ is a proper rational function in which the denominator is factorised as

$$a(z) = \prod_{j=1}^{r}(1 - z/\lambda_j)^{s_j}, \tag{7}$$

where $s_j$ is the multiplicity of the root $\lambda_j$, and where $\sum_j s_j = p$ is the degree of the polynomial. Then, the decomposition is

$$\frac{b(z)}{a(z)} = \sum_{j=1}^{r}\sum_{k=1}^{s_j}\frac{c_{j,k}}{(1 - z/\lambda_j)^k}; \tag{8}$$

and now there is the task of finding the series expansions of the partial fractions.

First, one may consider the case of a partial fraction that contains a distinct (unrepeated ) real root $\lambda$. The expansion is

$$\frac{c}{1 - z/\lambda} = c\{1 + z/\lambda + (z/\lambda)^2 + \cdots\}. \tag{9}$$

For this to converge for all $|z| \leq 1$, it is necessary and sufficient that $|\lambda| > 1$; and this is necessary and sufficient for the satisfaction of the BIBO condition.

Next, consider the case where $a(z)$ has a distinct pair of conjugate complex roots $\lambda$ and $\lambda^*$. These will come from a partial fraction with a quadratic denominator:

$$\frac{gz + d}{(1 - z/\lambda)(1 - z/\lambda^*)} = \frac{c}{1 - z/\lambda} + \frac{c^*}{1 - z/\lambda^*}. \tag{10}$$

It can be seen that $c = (g\lambda + d)/(1 - \lambda/\lambda^*)$ and $c^* = (g\lambda^* + d)/(1 - \lambda^*/\lambda)$ are also conjugate complex numbers.

The expansion of (9) applies to complex roots as well as to real roots:

$$\begin{aligned}\frac{c}{1 - z/\lambda} + \frac{c^*}{1 - z/\lambda^*} &= c\big\{1 + z/\lambda + (z/\lambda)^2 + \cdots\big\} \\ &\quad + c^*\big\{1 + z/\lambda^* + (z/\lambda^*)^2 + \cdots\big\} \\ &= \sum_{t=0}^{\infty} z^t(c\lambda^{-t} + c^*\lambda^{*-t}).\end{aligned} \tag{11}$$

The various complex quantities can be represented in terms of exponentials:

$$\begin{aligned} \lambda &= \kappa^{-1}e^{-i\omega}, \qquad & \lambda^* &= \kappa^{-1}e^{i\omega}, \\ c &= \rho e^{-i\theta}, & c^* &= \rho e^{i\theta}. \end{aligned} \tag{12}$$

Then, the generic term in the expansion becomes

$$\begin{aligned} z^t(c\lambda^{-t} + c^*\lambda^{*-t}) &= z^t\rho\kappa^t\big\{e^{i(\omega t-\theta)} + e^{-i(\omega t-\theta)}\big\} \\ &= z^t 2\rho\kappa^t\cos(\omega t-\theta). \end{aligned} \tag{13}$$

The expansion converges for all $|z| \leq 1$ if and only if $|\kappa| < 1$, which is a condition on the complex modulus of $\kappa$. But, $|\kappa| = |\lambda^{-1}| = |\lambda|^{-1}$; so it is confirmed that the necessary and sufficient condition for convergence is that $|\lambda| > 1$.

Finally, consider the case of a repeated root with a multiplicity of $s$. Then, a binomial expansion is available that gives

$$\frac{1}{(1-z/\lambda)^n} = 1 + n\frac{z}{\lambda} + \frac{n(n+1)}{2!}\left(\frac{z}{\lambda}\right)^2 + \frac{n(n+1)(n+2)}{3!}\left(\frac{z}{\lambda}\right)^3 + \cdots. \tag{14}$$

If $\lambda$ is real, then $|\lambda| > 1$ is the condition for convergence. If $\lambda$ is complex then it can be combined with the conjugate roots in the manner of (13) to create a trigonometric function and again, the the condition for convergence is that $|\lambda| > 1$.

The general conclusion is that the transfer function is stable if and only if all of the roots of the denominator polynomial $a(z)$, which are described as the poles of the transfer function, lie outside the unit circle in the complex plane.

It is helpful to represent the poles of the transfer function graphically by showing their locations within the complex plane together with the locations of the roots of the numerator polynomial, which are described as the zeros of the transfer function. It is more convenient to represent the poles and zeros of $b(z^{-1})/a(z^{-1})$, which are the reciprocals of those of $b(z)/a(z)$, since, for a stable and invertible transfer function, these must lie within the unit circle.

### The Frequency Reponse

It is of interest to consider the response of the transfer function to a simple sinusoidal signal. It is possible to represent a finite sequence as a sum of discretely sampled sine and cosine functions whose frequencies are integer multiples of a fundamental frequency that produces one cycle in the period spanned by the sequence.

It is also possible to represent an arbitrary stationary stochastic process, indexed by the doubly infinite set of integers, as a combination of an infinite number of sine and cosine functions whose frequencies range continuously in the interval $[0, \pi]$. It follows that the effect of a transfer function upon stationary signals can be characterised in terms of its effect upon the constituent sinusoidal functions.

Consider, therefore, the consequences of mapping the signal sequence $\{x_t = \cos(\omega t)\}$ through the transfer function with the coefficients $b_0, b_1, \ldots, b_q$. The output is

$$y(t) = \sum_{j=0}^{q} b_j \cos\bigl(\omega[t-j]\bigr). \tag{15}$$

The trigonometrical identity $\cos(A-B) = \cos A \cos B + \sin A \sin B$ enables us to write this as

$$\begin{aligned} y(t) &= \Bigl\{\sum_j b_j \cos(\omega j)\Bigr\} \cos(\omega t) + \Bigl\{\sum_j b_j \sin(\omega j)\Bigr\} \sin(\omega t) \\ &= \alpha \cos(\omega t) + \beta \sin(\omega t) = \rho \cos(\omega t - \theta). \end{aligned} \tag{16}$$

Here we have defined

$$\begin{aligned} &\alpha = \sum_{j=0}^{q} b_j \cos(\omega j), \qquad \beta = \sum_{j=0}^{q} b_j \sin(\omega j), \\ &\rho = \sqrt{\alpha^2 + \beta^2} \qquad \text{and} \qquad \theta = \tan^{-1}\Bigl(\frac{\beta}{\alpha}\Bigr). \end{aligned} \tag{17}$$

It can be seen from (8) that the effect of the transfer function upon the signal is twofold. First, there is a *gain effect* whereby the amplitude of the sinusoid is increased or diminished by a factor of $\rho$. Also, there is a *phase effect* whereby the peak of the sinusoid is displaced by a time delay of $\theta/\omega$ periods. The frequency of the output is the same as the frequency of the input. which is a fundamental feature of all linear dynamic systems.

Observe that the response of the transfer function to a sinusoid of a particular frequency is akin to the response of a bell to a tuning fork. It gives very limited information regarding the characteristics of the system. To obtain full information, it is necessary to excite the system over the whole range of frequencies.

**Spectral Densities**

In a discrete-time system, there is a problem of aliasing whereby frequencies (i.e. angular velocities) in excess of $\pi$ radians per sampling interval are confounded with frequencies within the interval $[0, \pi]$. To understand this, consider the case of a pure cosine wave of unit amplitude and zero phase whose frequency $\omega$ lies in the interval $\pi < \omega < 2\pi$. Let $\omega^* = 2\pi - \omega$. Then

$$\begin{aligned}\cos(\omega t) &= \cos\big\{(2\pi - \omega^*)t\big\}\\ &= \cos(2\pi)\cos(\omega^* t) + \sin(2\pi)\sin(\omega^* t)\\ &= \cos(\omega^* t);\end{aligned} \tag{18}$$

which indicates that $\omega$ and $\omega^*$ are observationally indistinguishable. Here, $\omega^* \in [0, \pi]$ is described as the alias of $\omega > \pi$.

This demonstration indicates that a discrete-time signal is band-limited in frequency to the Nyquist interval $[0, \pi]$. Moreover, any stationary stochastic process can be expressed as a weighted combination of the non-denumerable infinity of sines and cosines of which the frequencies lie in Nyquist interval. Thus, if $x_t$ is an element of such a process, then it can be represented by

$$x_t = \int_0^\pi \Big\{\cos(\omega t)dA(\omega) + \sin(\omega t)dB(\omega)\Big\}. \tag{19}$$

Here, $dA(\omega)$ and $dB(\omega)$ are the infinitesimal increments of stochastic functions defined on the frequency interval which are everywhere continuous but nowhere differentiable. Moreover, it is assumed that the increments $A(\omega)$ and $B(\omega)$ are uncorrelated with each other and with proceeding and succeeding increments.

The analysis of such stationary stochastic processes is facilitated by representing the trigonometrical functions in terms of complex exponentials. Thus

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{-i}{2}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}). \tag{20}$$

These enable equation (11) to be written as

$$\begin{aligned} x_t &= \int_0^\pi \left\{ \frac{(e^{i\omega t} + e^{-i\omega t})}{2} dA(\omega) - i\frac{(e^{i\omega t} - e^{-i\omega t})}{2} dB(\omega)\right\}\\ &= \int_0^\pi \left\{ e^{i\omega t}\frac{\{dA(\omega) - idB(\omega)\}}{2} + e^{-i\omega t}\frac{\{dA(\omega) + idB(\omega)\}}{2}\right\}.\end{aligned} \tag{21}$$

On defining

$$dZ(\omega) = \frac{dA(\omega) - idB(\omega)}{2} \quad \text{and} \quad dZ^*(\omega) = \frac{dA(\omega) + idB(\omega)}{2}$$

and extending the integral over the range $[-\pi, \pi]$, this becomes

$$x_t = \int_{-\pi}^{\pi} e^{i\omega t} dZ(\omega), \tag{22}$$

which is commonly described as the spectral representation of the process generating $y_t$.

The spectral density function of the process is the function $f(\omega)$ that is defined by

$$E\{dZ(\omega)dZ^*(\omega)\} = E\{dA^2(\omega) + dB^2(\omega)\} = f(\omega)d\omega. \tag{23}$$

The increment $f(\omega)d\omega$ is the power or the variance of the process that is attributable to the the elements in the frequency interval $[\omega, \omega + d\omega]$ and the integral of $f(\omega)$ over the frequency range $[-\pi, \pi]$ is the overall variance of the process. A white noise process $\{\varepsilon_t\}$ with a variance of $\sigma_\varepsilon^2$ has a uniform spectral density function $f_\varepsilon(\omega) = \sigma_\varepsilon^2/(2\pi)$.

Let the sequence $\{\psi_0, \psi_1, \ldots\}$ denote the impulse response of the transfer function. Then the effects of the transfer function upon the spectral elements of the process defined by (22) are shows by the equation

$$\begin{aligned} y_t &= \sum_j \mu_j x(t-j) = \sum_j \mu_j \left\{ \int_\omega e^{i\omega(t-j)} dZ_x(\omega) \right\} \\ &= \int_\omega e^{i\omega t} \left( \sum_j \mu_j e^{-i\omega j} \right) dZ_x(\omega). \end{aligned} \tag{23}$$

The effects are summarised by the complex-valued frequency-response function

$$\mu(\omega) = \sum \mu_j e^{-i\omega j} = |\mu(\omega)| e^{-i\theta(\omega)}, \tag{24}$$

The final expression, which is in polar form entails the following definitions:

$$\begin{aligned} |\mu(\omega)|^2 &= \left\{ \sum_{j=0}^{\infty} \mu_j \cos(\omega j) \right\}^2 + \left\{ \sum_{j=0}^{\infty} \mu_j \sin(\omega j) \right\}^2 \\ \theta(\omega) &= \arctan \left\{ \frac{\sum \mu_j \sin(\omega j)}{\sum \mu_j \cos(\omega j)} \right\}. \end{aligned} \tag{25}$$

These two components of the frequency response are the amplitude response or the gain $|\mu(\omega)|$ and the phase response $\theta(\omega)$.

The frequency response is just the discrete-time Fourier transform of the impulse response function. Equally, it is the $z$-transform $\psi(z^{-1}) = \sum_j \psi_j z^{-1}$ evaluated at the points $e^{i\omega}$ that lie on the unit circle in the complex plane.

As $\omega$ progresses from $-\pi$ to $\pi$, or, equally, as $z = e^{i\omega}$ travels around the unit circle, the frequency-response function defines a trajectory in the complex plane which becomes a closed contour when $\omega$ reaches $\pi$. The points on the trajectory are characterised by their polar co-ordinates. These are the modulus

$|\psi(\omega)|$, which is the length of the radius vector joining $\psi(\omega)$ to the origin, and the argument $\text{Arg}\{\psi(\omega)\} = -\theta(\omega)$ which is the (anticlockwise) angle in radians which the radius makes with the positive real axis.

The spectral density of the output function $f_y(\omega)$ of the filtered process $y(t)$ is given by

$$
\begin{aligned}
f_y(\omega)d\omega &= E\{dZ_y(\omega)dZ_y^*(\omega)\} \\
&= \mu(\omega)\mu^*(\omega)E\{dZ_x(\omega)dZ_x^*(\omega)\} \\
&= |\mu(\omega)|^2 f_x(\omega)d\omega.
\end{aligned}
\tag{26}
$$

## The Reparametrisation of Dynamic Models

In this section, we demonstrate a simple identity affecting polynomials in the lag operator and we show how this can be used in describing relationships between cointegrated time series. The same identity has been used to express an ARIMA process as the sum of a stationary stochastic process and an ordinary random walk. This expression is commonly know as the Beveridge–Nelson decomposition after its original proponents.

*A Polynomial Identity*

Let $\beta(z) = \beta_0 + \beta_1 z + \cdots + \beta_k z^k = \sum_{j=0}^{k} \beta_j z^j$ be a polynomial of degree $k$ in the argument $z$. We wish to show that this can be written in the following forms:

$$
\begin{aligned}
\beta(z) &= \beta(1) + \nabla(z)\gamma(z) \\
&= z^n\beta(1) + \nabla(z)\delta_n(z),
\end{aligned}
\tag{27}
$$

where $\nabla(z) = 1 - z$, where $0 \leq n \leq k$ and where $\gamma(z)$ and $\delta_n(z)$ are both polynomials of degree $k-1$. Also, $\beta(1)$ is the constant which is obtained by setting $z = 1$ in the polynomial $\beta(z)$.

To obtain the first expression on the RHS of (1), we divide $\beta(z)$ by $\nabla(z) = 1 - z$ to obtain a quotient of $\gamma(z)$ and a remainder of $\delta$:

$$
\beta(z) = \gamma(z)(1 - z) + \delta. \tag{28}
$$

Setting $z = 1$ in this equation gives

$$
\delta = \beta(1) = \beta_0 + \beta_1 + \cdots + \beta_k. \tag{29}
$$

This is an instance of the well-known remainder theorem of polynomial division. The coefficients of the quotient polynomial $\gamma(z)$ are given by

$$
\gamma_p = -\sum_{j=p+1}^{k} \beta_j, \quad \text{where} \quad p = 0, \ldots, k-1. \tag{30}
$$

There is a wide variety of ways in which these coefficients may de derived, including the familiar method of long division. Probably, the easiest way is via the method of synthetic division which may be illustrated by an example.

**Example.** Consider the case where $k = 3$. Then

$$\beta_0 + \beta_1 z + \beta_2 z^2 + \beta_3 z^3 = (\gamma_0 + \gamma_1 z + \gamma_2 z^2)(1 - z) + \delta. \tag{31}$$

By equating the coefficients associated with the same powers of $z$ on either side of the equation, we obtain the following identities:

$$\begin{aligned} \beta_3 &= -\gamma_2 \\ \beta_2 &= -\gamma_1 + \gamma_2 \\ \beta_1 &= -\gamma_0 + \gamma_1 \\ \beta_0 &= \delta + \gamma_0. \end{aligned} \tag{32}$$

These can be rearranged to give

$$\begin{aligned} \gamma_2 &= -\beta_3 \\ \gamma_1 &= -\beta_2 + \gamma_2 &&= -(\beta_2 + \beta_3) \\ \gamma_0 &= -\beta_1 + \gamma_1 &&= -(\beta_1 + \beta_2 + \beta_3) \\ \delta &= \beta_0 - \gamma_0 &&= \beta_0 + \beta_1 + \beta_2 + \beta_3. \end{aligned} \tag{33}$$

To obtain the second expression on the RHS of (27), consider the identity

$$1 = z^n + \nabla(z)(1 + z + \cdots + z^{n-1}), \tag{34}$$

where $1 < n \leq k$. Multiplying both sides by $\beta(1)$ gives

$$\beta(1) = z^n \beta(1) + \nabla(z)\{1 + z + \cdots + z^{n-1}\}\beta(1), \tag{35}$$

On substituting this expression into the first equation of (27) and on defining

$$\delta_n(z) = \gamma(z) + \{1 + z + \cdots + z^{n-1}\}\beta(1), \tag{36}$$

we can write

$$\beta(z) = z^n \beta(1) + \nabla(z)\delta_n(z). \tag{37}$$

This is a general expression which covers both equations of (27), since setting $n = 0$ reduces it to the first equation.

A leading instance is obtained by setting $n = 1$. In that case, equation (35) gives $\beta(1) = z\beta(1) + \nabla(z)\beta(1)$. Substituting this into $\beta(z) = \beta(1) + \nabla(z)\gamma(z)$ gives the following specialisation of equation (11):

$$\begin{aligned} \beta(z) &= z\beta(1) + \nabla(z)\{\gamma(z) + \beta(1)\} \\ &= z\beta(1) + \nabla(z)\delta_1(z). \end{aligned} \tag{38}$$

By comparing coefficients associated with the same powers of $z$ on both sides of the equation, it can be seen that the constant term of the polynomial $\delta_1(z)$ is just $\beta_0$. Reference to (30) and (34) confirms this result.

*Reparametrisation of a Distributed Lag Model*

Consider a distributed-lag model of the form

$$y(t) = \beta_0 x(t) + \beta_1 x(t-1) + \cdots + \beta_k x(t-k) + \varepsilon(t). \tag{39}$$

This can be written is summary notation as

$$y(t) = \beta(L)x(t) + \varepsilon(t), \tag{40}$$

where $\beta(L) = \beta_0 + \beta_1 L + \cdots + \beta_k L^k$ is a polynomial in the lag operator $L$.

Using the basic identity from (27), we can set $\beta(L) = \beta(1) + \nabla\delta(L)$. Then taking $y(t-1)$ from both sides of the resulting equation gives

$$\nabla y(t) = \{\beta(1)x(t) - y(t-1)\} + \delta(L)\nabla x(t) + \varepsilon(t), \tag{41}$$

which is an error-correction formulation of equation (40).

*Reparametrisation of an Autoregressive Distributed Lag Model*

Now consider an equation in the form of

$$y(t) = \phi_1 y(t-1) + \cdots + \phi_p y(t-p) + \beta_0 x(t) + \cdots + \beta_k x(t-k) + \varepsilon(t), \tag{42}$$

which can be written is summary notation as

$$\alpha(L)y(t) = \beta(L)x(t) + \varepsilon(t), \tag{43}$$

with $\alpha(L) = \alpha_0 + \alpha_1 L + \cdots + \alpha_p L^p = 1 - \phi_1 L - \cdots - \phi_p L^p$. On setting $\alpha(L) = \alpha(1)L + \theta_1(L)\nabla$ and $\beta(L) = \beta(1)L + \delta_1(L)\nabla$, this can be rewritten as

$$\{\alpha(1)L + \theta_1(L)\nabla\}y(t) = \{\beta(1)L + \delta_1(L)\nabla\}x(t) + \varepsilon(t), \tag{44}$$

where the leading element of $\theta_1(L)$ is $\alpha_0 = 1$. Define

$$\rho(L) = \rho_1 L + \rho_2 L^2 + \cdots + \rho_p L^p = 1 - \theta_1(L). \tag{45}$$

Then equation (44) can be rearranged to give

$$\begin{aligned}\nabla y(t) &= \{\beta(1)Lx(t) - \alpha(1)Ly(t)\} + \delta_1(L)\nabla x(t) + \rho(L)\nabla y(t) + \varepsilon(t)\\ &= \lambda\{\gamma x(t-1) - y(t-1)\} + \delta_1(L)\nabla x(t) + \rho(L)\nabla y(t) + \varepsilon(t),\end{aligned} \tag{46}$$

where $\lambda = \alpha(1)$ is the so-called adjustment parameter and where $\gamma = \beta(1)/\alpha(1)$ is the steady-state gain of the rational transfer function $\beta(L)/\alpha(L)$. The term $\gamma x(t-1) - y(t-1)$ is described as the equilibrium error; and the value of the error will tend to zero if a steady state is maintained by $x(t)$ and if there are no disturbances. Equation (46) is the classical form of the error-correction equation.

*The Beveridge–Nelson Decomposition*

Consider an ARIMA model which is represented by the equation

$$\alpha(L)\nabla y(t) = \mu(L)\varepsilon(t) \tag{47}$$

Dividing both sides by $\alpha(L)$ gives

$$\begin{aligned}\nabla y(t) &= \frac{\mu(L)}{\alpha(L)}\varepsilon(t)\\ &= \psi(L)\varepsilon(t),\end{aligned} \tag{48}$$

where $\psi(L)$ stands for the power series expansion of the rational function. If the coefficients of this expansion form an absolutely summable sequence, then $\psi(L) = \psi(1) + \nabla\lambda(L)$, which a decomposition in the form of equation (27). Then

$$\begin{aligned}\nabla y(t) &= \psi(1)\varepsilon(t) + \lambda(L)\nabla\varepsilon(t)\\ &= \nabla v(t) + \nabla w(t).\end{aligned} \tag{49}$$

Here, $v(t) = \psi(1)\nabla^{-1}\varepsilon(t)$ in first term on the RHS stands for a random walk, whereas $w(t) = \lambda(L)\varepsilon(t)$ stands for a stationary stochastic process.

For another perspective on this decomposition, we may consider the following partial-fraction decompostion:

$$\frac{\mu(z)}{\alpha(z)\nabla(z)} = \frac{\gamma(z)}{\alpha(z)} + \frac{\delta}{\nabla(z)}. \tag{50}$$

Multiplying both sides by $\nabla(z)$ gives

$$\frac{\mu(z)}{\alpha(z)} = \frac{\nabla(z)\gamma(z)}{\alpha(z)} + \delta, \tag{51}$$

whence setting $z = 1$ gives $\delta = \mu(1)/\alpha(1)$. Thereafter, we can find $\gamma(z) = \{\mu(z) - \delta\alpha(z)\}/\nabla(z)$