

ECONOMETRIC THEORY: Exercise 1 (Tutorial)

Matrix Algebra and Manpower Planning

Matrix Multiplication in Microsoft Excel

There is a limited set of matrix operations which may be performed in *Excel* with the use of predefined commands. The following commands are available in *Version 4.0*: *MMULT*, *MINVERSE*, *TRANSPOSE* and *MDETERM*. In this exercise, we shall use only the first of these. For practice, we may multiply the matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} 7 & 9 & 11 \\ 8 & 10 & 12 \end{bmatrix}$$

to form the product

$$AB = C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \begin{bmatrix} 39 & 49 & 59 \\ 54 & 68 & 82 \\ 69 & 87 & 105 \end{bmatrix}.$$

To form the generic element c_{ij} in the i th row and the j th column of $C = AB$, we take the elements of the i th row of A , which is $a_{i.} = [a_{i1}, a_{i2}]$, and we combine them with the elements of the j th column of B , which is $b_{.j} = [b_{1j}, b_{2j}]'$, to give $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j}$.

Notice that the j th column of B has been written as a row followed by a prime $\{ '\}$ which indicates that the row is to be transposed to form a column. We shall often employ this space-saving device for representing columns.

Figure 1 shows the basic format and procedure for multiplying matrices. There are three steps:

- (i) Insert the elements of the matrices A and B in convenient arrays of cells. In the figure, the elements of A are in the range (B5:C7) and those of B are in the range (F5:H6).
- (ii) Select a output range in which the results are to be written. In the figure, the output range is (B11:D13).
- (iii) Type the formula =MMULT(B5:C7, F5:H6) and then press <Ctrl>-<Shift>.

When the final command is issued the entry in the formula bar becomes {=MMULT(B5:C7, F5:H6)} and the results appear in the output range.

The other matrix commands are used in the same way. References to them may be found found via the following sequence of menu commands: *Windows-Help-Reference-Worksheet Functions -Math & Trig Functions*.

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	A	B	C	D	E	F	G	H
1	Basic Matrix Multiplication: =MMULT(array1, array2)							
2								
3	First Range to Multiply				Second Range to Multiply			
4	(B5:C7)				(F5:H6)			
5		1	4			7	9	11
6	A =	2	5		B =	8	10	12
7		3	6					
8								
9	Output Range				These results may be obtained			
10	(B11:D13)				using non-matrix methods as			
11		39	49	59	is indicated in the two examples			
12	C =	54	68	82				
13		69	87	105				
14								
15	Example: To form the product				Example: To form the product			
16	in the 1st row and 1st column of				in the 3rd row and 3rd column of			
17	the matrix C which is cell B11:				the matrix C which is cell D13:			
18	Formula: =(B5*F5 + C5*F6)				Formula: =(B7*H5 + C7*H6)			
19	Numbers: 1*7 + 4*8 = 39				Numbers: 3*11 + 6*12 = 105			

Figure 1. Matrix multiplication in *Excel 4.0*.

Manpower Planning

The object of manpower planning is to ensure that there are appropriate numbers of workers in each category to meet the needs of an organisation. To achieve this, the planners must take account of the variety of ways in which persons may pass through the hierarchy of the organisation as time elapses. There are three factors to be taken into account:

- (i) The effects of promotions from one category to another,
- (ii) The effects of retirement and wastage,
- (iii) The effects of recruitment.

The following table represents, for each of four categories of employment, indexed by $i = 1, \dots, 4$, the number of incumbents $x_{i,t-1}$ at the start of a calendar year who have survived from the previous year, the number r_{it} recruited throughout the year, the number leaving which is q_{it} , and the number x_{it} carried over to the next year.

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<i>Category</i>	<i>Incumbents</i>	<i>Recruits</i>	<i>Leavers</i>	<i>Survivors</i>
1 : Executives	$x_{1,t-1} = 100$	$r_{1t} = 5$	$q_{1t} = 20$	$x_{1t} = 105$
2 : Managers	$x_{2,t-1} = 200$	$r_{2t} = 10$	$q_{2t} = 40$	$x_{2t} = 230$
3 : Supervisors	$x_{3,t-1} = 800$	$r_{3t} = 100$	$q_{3t} = 240$	$x_{3t} = 780$
4 : Assistants	$x_{4,t-1} = 2000$	$r_{4t} = 500$	$q_{4t} = 400$	$x_{4t} = 1900$

Table 1. The overall employment statistics for the organisation in the year t .

The discrepancies in the figures are due to internal promotions. Detailed investigation of the organisation reveals the following pattern of promotion which we may regard as typical for all years:

<i>Promotees</i>	<i>Executives</i>	<i>Managers</i>	<i>Supervisors</i>	<i>Assistants</i>
1: Executives	80	0	0	0
2 : Managers	20	140	0	0
3 : Supervisors	0	80	480	0
4 : Assistants	0	0	200	1400

Table 2. The promotion within the organisation during the year t .

By reading along the second row for example, we find that, of the 200 managers on the staff at the start of the year, 140 remained *in situ*, whereas 20 were promoted to the grade of Executive. The remaining 40 left the organisation, as is shown in Table 1.

Let $h_{ij,t}$ denote the numbers of workers promoted from the j th category to the i th category. Then the coefficient $p_{ij} = h_{ij,t}/x_{j,t-1}$ is the proportion of the workers in the j th category who receive this promotion. If we regard these proportions as invariant from year to year, then the number of employees in the i th category at the end of the year t will be determined by the following equation:

$$x_{it} = p_{i1}x_{1,t-1} + p_{i2}x_{2,t-1} + p_{i3}x_{3,t-1} + p_{i4}x_{4,t-1} + r_{it}$$

The four equations for $i := 1, \dots, 4$ may be collected in a matrix format:

$$\begin{bmatrix} x_{1t} \\ x_{2t} \\ x_{3t} \\ x_{4t} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix} \begin{bmatrix} x_{1,t-1} \\ x_{2,t-1} \\ x_{3,t-1} \\ x_{4,t-1} \end{bmatrix} + \begin{bmatrix} r_{1t} \\ r_{2t} \\ r_{3t} \\ r_{4t} \end{bmatrix}.$$

This can be represented in summary notation as

$$x_t = Px_{t-1} + r_t.$$

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The equation gives rise to a recursive extrapolation. Thus, for one year ahead, we have

$$\begin{aligned}x_{t+1} &= Px_t + r_{t+1} \\ &= P\{Px_{t-1} + r_t\} + r_{t+1} \\ &= PPx_{t-1} + \{Pr_t + r_{t+1}\};\end{aligned}$$

and therefore, in general, we have

$$x_{t+n} = P^{n+1}x_{t-1} + \{P^n r_t + P^{n-1}r_{t+1} + \cdots + r_{t+n}\}$$

for n years ahead, where $P^n = P \times P \times \cdots \times P$ stands for the n -fold multiplication of the matrix P by itself.

It is clear from the context of the problem that, as n increases, the significance of the term $P^{n+1}x_{t-1}$ will diminish; for this corresponds to the notion that, as time passes, the longer-serving members of the organisation will retire or waste away. More formally, it can be asserted that, as n increases, the significance of the initial conditions, which are the elements of the vector x_{t-1} , diminishes.

For an exercise, you are asked to perform the following tasks:

- (i) Using tables 1 and 2, calculate the elements of the matrix P for the year t in question.
- (ii) Assuming that the matrix P stays constant and that the pattern of recruitments of the year t is repeated indefinitely, calculate the numbers in each of the categories of employment in the ensuing 5 years.
- (ii) Use the facilities of *Excel 4.0* to construct a bar chart to reveal the distributions of the employees amongst the categories over the six years, $t, \dots, t+5$; and attempt, in some way, to characterise the long-term tendency.