14 : APPENDIX

Polynomials and Difference Equations

A time-series model is one which postulates a relationship amongst a number of temporal sequences or time series. For example, we have the regression model

$$y(t) = x(t)\beta + \varepsilon(t) \tag{1}$$

where $x(t)$ is an observable sequence indexed by the time subscript t and $\varepsilon(t)$ is an unobservable sequence of independently and identically distributed random variables.

A more general model, which we shall call the general temporal regression model, is one which postulates a relationship comprising any number of consecutive elements of $x(t)$, $y(t)$ and $\varepsilon(t)$. Thus we have

$$\sum_{i=0}^{p} \alpha_i y(t-i) = \sum_{i=0}^{k} \beta_i x(t-i) + \sum_{i=0}^{q} \mu_i \varepsilon(t-i) \tag{2}$$

Any of the sums in this expression can be infinite, but if the model is to be viable, the sequences of coefficients $\{\alpha_i\}$, $\{\beta_i\}$ and $\{\mu_i\}$ can depend on only a limited number of parameters.

We are particularly interested in a number of specialisations of the model. The model represented by the equation

$$y(t) = \beta x(t-i) + \sum_{i=0}^{q} \mu_i \varepsilon(t-i) \tag{3}$$

is described as a regression model with a serially correlated disturbance sequence $\eta(t) = \sum \mu_i \varepsilon(t-i)$. If this sum is finite, then $\eta(t)$ is called a moving average process. The model represented by

$$\sum_{i=0}^{p} \alpha_i y(t-i) = \beta x(t) + \varepsilon(t) \tag{4}$$

is described as an autoregressive regression model. The equation

$$y(t) = \sum_{i=0} \beta_i x(t-i) + \varepsilon(t) \tag{5}$$

represents the distributed-lag regression model.

The foregoing models are all termed regression models by virtue of the inclusion of the observable explanatory sequence $x(t)$. When $x(t)$ is deleted, we obtain the simpler unconditional linear stochastic models. Thus the equation

$$y(t) = \sum_{i=0}^{q} \mu_i \varepsilon(t-i), \tag{6}$$

where the sum is finite, represents a moving average process; whereas

$$\sum_{i=0}^{p} \alpha_i y(t-i) = \varepsilon(t), \tag{7}$$

again with a finite sum, represents an autoregressive process. The equation

$$\sum_{i=0}^{p} \alpha_i y(t-i) = \sum_{i=0}^{q} \mu_i \varepsilon(t-i) \tag{8}$$

represents an autoregressive-moving average process.

The Algebra of the Lag Operator

A sequence $x(t)$ is any function mapping from the set of integers $\mathcal{Z} = \{0, \pm 1, \pm 2, \ldots\}$ to the real line. If the set of integers represents a set of dates separated by unit intervals, then we say that $x(t)$ is a temporal sequence or time series.

The set of all time series represents a vector space, and we can define various linear transformations or operators over the space. For example, we define the lag operator by

$$Lx(t) = x(t-1). \tag{9}$$

Now, $L\{Lx(t)\} = Lx(t-1) = x(t-2)$; so it makes sense to define L^2 by $L^2x(t) = x(t-2)$. More generally, $L^kx(t) = x(t-k)$ and, likewise, $L^{-k}x(t) = x(t+k)$. Other operators are the difference operator $\nabla = I - L$ which has the effect that

$$\nabla x(t) = x(t) - x(t-1), \tag{10}$$

the forward-difference operator $\Delta = L^{-1} - I$, and the summation operator $S = (I - L)^{-1} = \{I + L + L^2 + \cdots\}$ which has the effect that

$$Sx(t) = \sum_{i=0}^{\infty} x(t-i). \tag{11}$$

In general, we can define polynomials of the lag operator of the form $p(L) = p_0 + p_1 L + p_2 L^2 + \cdots + p_n L^n = \sum p_i L^i$ having the effect that

$$\begin{aligned} p(L)x(t) &= p_0 x(t) + p_1 x(t-1) + \cdots + p_n x(t-n) \\ &= \sum_{i=0}^{n} p_i x(t-i). \end{aligned} \tag{12}$$

The advantage which comes from defining polynomials in the lag operator stems from the fact that they are isomorphic to the set of ordinary algebraic polynomials. Thus we can rely upon what we know about ordinary polynomials to treat problems concerning lag-operator polynomials.

Algebraic Polynomials

Consider the equation $\alpha_0 + \alpha_1 z + \alpha_2 z^2 = 0$. On dividing the equation by α_2, we can factorise it as $(z - \lambda_1)(z - \lambda_2)$ where λ_1, λ_2 are the roots of the equation that are given by the formula

$$\lambda = \frac{-\alpha_1 \pm \sqrt{\alpha_1^2 - 4\alpha_2\alpha_0}}{2\alpha_2}. \tag{13}$$

If $\alpha_1^2 \geq 4\alpha_2\alpha_0$, then the roots λ_1, λ_2 are real. If $\alpha_1^2 = 4\alpha_2\alpha_0$, then $\lambda_1 = \lambda_2$. If $\alpha_1^2 \leq 4\alpha_2\alpha_0$, then the roots are the conjugate complex numbers $\lambda = \alpha + i\beta$, $\lambda^* = \alpha - i\beta$ where $i = \sqrt{-1}$.

Complex Roots

There are three alternative ways of representing the conjugate complex numbers λ and λ^* :

$$\begin{aligned} \lambda &= \alpha + i\beta = \rho(\cos\theta + i\sin\theta) = \rho e^{i\theta}, \\ \lambda^* &= \alpha - i\beta = \rho(\cos\theta - i\sin\theta) = \rho e^{-i\theta}, \end{aligned} \tag{14}$$

where

$$\rho = \sqrt{(\alpha^2 + \beta^2)} \qquad \text{and} \qquad \theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right). \tag{15}$$

These are called, respectively, the Cartesian form, the trigonometrical form and the exponential form.

The polar representation is understood by considering the Argand diagram:

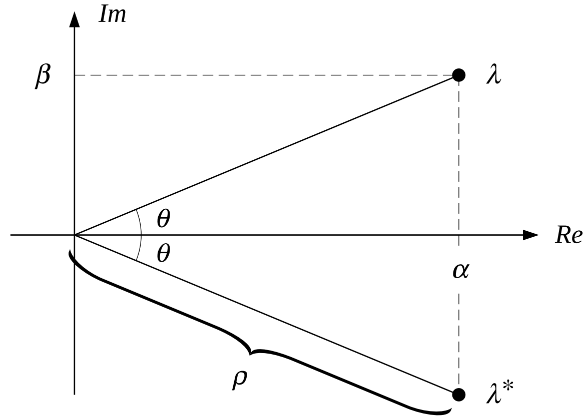

Figure 1. The Argand Diagram showing a complex number $\lambda = \alpha + i\beta$ and its conjugate $\lambda^* = \alpha - i\beta$.

The exponential form is understood by considering the following series expansions of $\cos\theta$ and $i\sin\theta$ about the point $\theta = 0$:

$$
\begin{aligned}
\cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots \\
i\sin\theta &= i\theta - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} - \frac{i\theta^7}{7!} + \cdots
\end{aligned} \tag{16}
$$

Adding these gives us Euler's equation:

$$
\begin{aligned}
\cos\theta + i\sin\theta &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots \\
&= e^{i\theta}.
\end{aligned} \tag{17}
$$

Likewise, by subtraction, we get

$$\cos\theta - i\sin\theta = e^{-i\theta}. \tag{18}$$

It follows readily from (17) and (18) that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{19}$$

and

$$\sin\theta = \frac{-i(e^{i\theta} - e^{-i\theta})}{2} = \frac{e^{i\theta} - e^{-i\theta}}{2i}. \tag{20}$$

The n-th Order Polynomial

Now consider the general equation of the nth order:

$$\alpha_0 + \alpha_1 z + \alpha_2 z^2 + \cdots + \alpha_n z^n = 0. \tag{21}$$

On dividing by α_n, we can factorise this as

$$(z - \lambda_1)(z - \lambda_2)\cdots(z - \lambda_n) = 0, \tag{22}$$

where some of the roots may be real and others may be complex. The complex roots come in conjugate pairs, so that, if $\lambda = \alpha + i\beta$ is a complex root, then there is a corresponding root $\lambda^* = \alpha - i\beta$ such that the product $(z-\lambda)(z-\lambda^*) = z^2 + 2\alpha z + (\alpha^2 + \beta^2)$ is real and quadratic. When we multiply the n factors together, we obtain the expansion

$$0 = z^n - \sum_i \lambda_i z^{n-1} + \sum_i \sum_j \lambda_i \lambda_j z^{n-2} - \cdots (-1)^n \lambda_1 \lambda_2 \cdots \lambda_n. \tag{23}$$

We can compare this with the expression $(\alpha_0/\alpha_n) + (\alpha_1/\alpha_n)z + \cdots + z^n = 0$. By equating coefficients of the two expressions, we find that $(\alpha_0/\alpha_n) = (-1)^n \prod \lambda_i$ or, equivalently,

$$\alpha_n = \alpha_0 \prod_{i=1}^{n} (-\lambda_i)^{-1}. \tag{24}$$

Thus we can express the polynomial in any of the following forms:

$$\begin{aligned} \sum \alpha_i z^i &= \alpha_n \prod (z - \lambda_i) \\ &= \alpha_0 \prod (-\lambda_i)^{-1} \prod (z - \lambda_i) \\ &= \alpha_0 \prod \left(1 - \frac{z}{\lambda_i}\right). \end{aligned} \tag{25}$$

We should also note that, if λ is a root of the equation $\sum \alpha_i z^i = 0$, then $\mu = 1/\lambda$ is a root of the equation $\sum \alpha_i z^{n-i} = 0$. This follows since

$\sum \alpha_i \mu^{n-i} = \mu_n \sum \alpha_i \mu^{-i} = 0$ implies that $\sum \alpha_i \mu^{-i} = \sum \alpha_i \lambda_i = 0$. Confusion can arise from not knowing which of the two equations one is dealing with.

Rational Functions of Polynomials

If $\delta(z)$ and $\gamma(z)$ are polynomial functions of x of degrees n and m respectively with $n < m$, then the ratio $\delta(z)/\gamma(z)$ is described as a proper rational polynomial. We shall often encounter expressions of the form

$$y(t) = \frac{\delta(L)}{\gamma(L)} x(t).$$

For this to have a meaningful interpretation in the context of a time-series model, we normally require that $y(t)$ should be a bounded sequence whenever $x(t)$ is bounded. The necessary and sufficient condition for the boundedness of $y(t)$, in that case, is that the series expansion of $\delta(z)/\gamma(z)$ should be convergent whenever $|z| \leq 1$. We can determine whether or not the sequence will converge by expressing the ratio $\delta(z)/\gamma(z)$ as a sum of partial fractions. The basic result is as follows:

(26) If $\delta(z)/\gamma(z) = \delta(z)/\{\gamma_1(z)\gamma_2(z)\}$ is a proper rational fraction, and if $\gamma_1(z)$ and $\gamma_2(z)$ have no common factor, then the fraction can be uniquely expressed as

$$\frac{\delta(z)}{\gamma(z)} = \frac{\delta_1(z)}{\gamma_1(z)} + \frac{\delta_2(z)}{\gamma_2(z)},$$

where $\delta_1(z)/\gamma_1(z)$ and $\delta_2(z)/\gamma_2(z)$ are proper rational fractions.

Imagine that $\gamma(z) = \prod(1 - z/\lambda_i)$. Then repeated applications of this basic result enables us to write

$$\delta(z) = \frac{\kappa_1}{1 - z/\lambda_1} + \frac{\kappa_2}{1 - z/\lambda_2} + \cdots + \frac{\kappa_n}{1 - z/\lambda_n}. \tag{27}$$

By adding the terms on the RHS, we find an expression with a numerator of order $n - 1$. By equating the terms of the numerator with the terms of $\delta(z)$, we can find the values $\kappa_1, \kappa_2, \ldots, \kappa_n$.

Consider, for example,

$$\begin{aligned} \frac{3x}{1 + x - 2x^2} &= \frac{3x}{(1 - x)(1 + 2x)} \\ &= \frac{\kappa_2}{1 - x} + \frac{\kappa_2}{1 + 2x} \\ &= \frac{\kappa_1(1 + 2x) + \kappa_2(1 - x)}{(1 - x)(1 + 2x)}. \end{aligned} \tag{28}$$

Equating the terms of the numerator gives

$$3x = (2\kappa_1 - \kappa_2)x + (\kappa_1 + \kappa_2), \tag{29}$$

so that $\kappa_2 = -\kappa_1$ which gives $3 = (2\kappa_1 - \kappa_2) = 3\kappa_1$; and thus we have $\kappa_1 = 1$, $\kappa_2 = -1$.

The conditions for the convergence of the expansion of $\delta(z)/\gamma(z)$ are straightforward. For the rational function converges if and only if the expansion of each of its partial fractions converges. For the expansion

$$\frac{\kappa}{(1 - z/\lambda)} = \kappa\Big\{1 + z/\lambda + (z/\lambda)^2 + \cdots\Big\} \tag{30}$$

to converge when $|z| \leq 1$, it is necessary and sufficient that $|\lambda| > 1$.

Linear Difference Equations

An nth-order linear difference equation is a relationship amongst $n + 1$ consecutive elements of a sequence $x(t)$ of the form

$$\alpha_0 x(t) + \alpha_1 x(t) + \cdots + \alpha_n x(t - n) = u(t), \tag{31}$$

where $u(t)$ is some specified sequence. We can also write this as

$$\alpha(L)x(t) = u(t), \tag{32}$$

where $\alpha(L) = \alpha_0 + \alpha_1 L + \alpha_2 L^2 + \cdots + \alpha_n L^n$. If we are given n consecutive values of $x(t)$, say $x_1, x_2, \ldots, x_n$, then we can use this relationship to find the succeeding value x_{n+1}. In this way, so long as $u(t)$ is fully specified, we can generate all the succeeding elements of the sequence. Likewise, we can generate all the values of the sequence prior to $t = 1$; and thus, in effect, we can deduce the function $x(t)$ from the difference equation. However, instead of a recursive solution, we usually seek an analytic expression for $x(t)$.

The function $x(t; c)$, expressing the analytic solution, will comprise a set of n constants in $c = [c_1, c_2, \ldots, c_n]'$ which can only be determined once we are given a set of n consecutive values of $x(t)$ which are called initial conditions. The general analytic solution of the equation $\alpha(L)x(t) = u(t)$ is expressed as $x(t; c) = y(t; c) + z(t)$, where $y(t)$ is the general solution of the homogeneous equation $\alpha(L)y(t) = 0$, and $z(t) = \alpha^{-1}(L)u(t)$ is called a particular solution of the inhomogeneous equation.

We solve the difference equation in three steps. First, we find the general solution of the homogeneous equation. Next, we find the particular solution $z(t)$ which embodies no unknown quantities. Finally, we use the n initial values of x to determine the constants $c_1, c_2, \ldots, c_n$. We shall discuss in detail only the solution of the homogeneous equation.

Solution of the Homogeneous Difference Equation

If λ_j is a root of the equation $\alpha(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \cdots + \alpha_n z^n = 0$ such that $\alpha(\lambda_j) = 0$, then $y_j(t) = (1/\lambda_j)^t$ is a solution of the equation $\alpha(L)y(t) = 0$. We can see this by considering the expression

$$\begin{aligned}
\alpha(L)\left(\frac{1}{\lambda_j}\right)^t &= (\alpha_0 + \alpha_1 L + \cdots + \alpha_n L^n)\left(\frac{1}{\lambda_j}\right)^t \\
&= \alpha_0\left(\frac{1}{\lambda_j}\right)^t + \alpha_1\left(\frac{1}{\lambda_j}\right)^{t-1} + \cdots + \alpha_n\left(\frac{1}{\lambda_j}\right)^{t-n} \\
&= (\alpha_0 + \alpha_1\lambda_j + \cdots + \alpha_n\lambda_j^n)\left(\frac{1}{\lambda_j}\right)^t \\
&= \alpha(\lambda_j)\left(\frac{1}{\lambda_j}\right)^t.
\end{aligned} \tag{33}$$

Alternatively, consider the factorisation $\alpha(L) = \alpha_0\prod_i(1 - L/\lambda_j)$. Within this product, we have the term $1 - L/\lambda_j$ and, clearly, since

$$\left(1 - \frac{L}{\lambda_j}\right)\left(\frac{1}{\lambda_j}\right)^t = \left(\frac{1}{\lambda_j}\right)^t - \left(\frac{1}{\lambda_j}\right)^t = 0,$$

we must have $\alpha(L)(1\lambda_j)^t = 0$.

The general solution, in the case where $\alpha(L) = 0$ has distinct real roots, is given by

$$y(t;c) = c_1\left(\frac{1}{\lambda_1}\right)^t + c_2\left(\frac{1}{\lambda_2}\right)^t + \cdots + c_n\left(\frac{1}{\lambda_n}\right)^t, \tag{34}$$

where $c_1, c_2, \ldots, c_n$ are the constants which are determined by the initial conditions.

In the case where two roots coincide at a value of λ, the equation $\alpha(L)y(t) = 0$ has the solutions $y_1(t) = (1/\lambda)^t$ and $y_2(t) = t(1/\lambda)^t$. To show this, let us extract the term $(1 - L/\lambda)^2$ from the factorisation $\alpha(L) = \alpha_0\prod_i(1 - L/\lambda_j)$ given under (25). Then, according to the previous argument, we have $(1 - L/\lambda)^2(1/\lambda)^t = 0$, but, also, we have

$$\begin{aligned}
\left(1 - \frac{L}{\lambda}\right)^2 t\left(\frac{1}{\lambda}\right)^t &= \left(1 - \frac{2L}{\lambda} + \frac{L^2}{\lambda^2}\right)t\left(\frac{1}{\lambda}\right)^t \\
&= t\left(\frac{1}{\lambda}\right)t - 2(t-1)\left(\frac{1}{\lambda}\right)^t + (t-2)\left(\frac{1}{\lambda}\right)^t = 0.
\end{aligned} \tag{35}$$

In general, if there are r repeated roots, then $(1/\lambda)^t$, $t(1/\lambda)^t$, $t^2(1/\lambda)^t$, $\ldots$, $t^{r-1}(1/\lambda)^t$ are all solutions to the equation $\alpha(L)y(t) = 0$.

The 2nd-order Difference equation with Complex Roots

Imagine that the 2nd-order equation $\alpha(L)y(t) = \alpha_0 y(t) + \alpha_1 y(t-1) + \alpha_2 y(t-2) = 0$ is such that $\alpha(z) = 0$ has complex roots $\lambda = 1/\mu$ and $\lambda^* = 1/\mu^*$. Let us write

$$\begin{aligned} \mu &= \gamma + i\delta = \kappa(\cos\omega + i\sin\omega) = \kappa e^{i\omega}, \\ \mu^* &= \gamma - i\delta = \kappa(\cos\omega - i\sin\omega) = \kappa e^{-i\omega}. \end{aligned} \tag{36}$$

These will appear in a general solution of the difference equation of the form of

$$y(t) = c\mu^t + c^*(\mu^*)^t. \tag{37}$$

This is a real-valued sequence; and, since a real term must equal its own conjugate, we require c and c^* to be conjugate numbers of the form

$$\begin{aligned} c^* &= \rho(\cos\theta + i\sin\theta) = \rho e^{i\theta}, \\ c &= \rho(\cos\theta - i\sin\theta) = \rho e^{-i\theta}. \end{aligned} \tag{38}$$

Thus we have

$$\begin{aligned} c\mu^t + c^*(\mu^*)^t &= \rho e^{-i\theta}(\kappa e^{i\omega})^t + \rho e^{i\theta}(\kappa e^{-i\omega})^t \\ &= \rho\kappa^t\Big\{e^{i(\omega t-\theta)} + e^{-i(\omega t-\theta)}\Big\} \\ &= 2\rho\kappa^t\cos(\omega t-\theta). \end{aligned} \tag{39}$$

To analyse the final expression, consider first the factor $\cos(\omega t - \theta)$. This is a displaced cosine wave. The value ω, which is a number of radians per unit period, is called the angular velocity or the angular frequency of the wave. The value $f = \omega/2\pi$ is its frequency in cycles per unit period. The duration of one cycle, also called the period, is $r = 2\pi/\omega$.

The term θ is called the phase displacement of the cosine wave, and it serves to shift the cosine function along the axis of t so that the peak occurs at the value of $t = \theta/\omega$ instead of at $t = 0$.

Next consider the term κ^t wherein $\kappa = \surd(\gamma^2 + \delta^2)$ is the modulus of the complex roots. When κ has a value of less than unity, it becomes a damping factor which serves to attenuate the cosine wave as t increases.

Finally, the factor 2ρ represents the initial amplitude of the cosine wave which is the value that it assumes when $t = 0$. Since ρ is just the modulus of the values c and c^*, this amplitude reflects the initial conditions.