4 : APPENDIX

The Geometry of the Plane

This appendix sets forth some algebraic geometry which can be applied to the problems of chapter 2. It culminates in a derivation of the errors-in-variables estimator according to the principle of least squares.

Vectors, points and line segments

A point in the plane is primarily a geometric object; but, if we introduce a coordinate system, then it may be described in terms of an ordered pair of numbers.

In constructing a coordinate system, it is usually convenient to introduce two perpendicular axes and to use the same scale of measurement on both axes. The point of intersection of these axes is called the origin and it is denoted by 0. The point on the first axis at a unit distance from the origin 0 is denoted by e_1 and the point on the second axis at a unit distance from 0 is denoted by e_2.

An arbitrary point a in the plane can be represented by its coordinates a_1 and a_2 relative to these axes. The coordinates are obtained by the perpendicular projections of the point onto the axes. If we are prepared to identify the point with its coordinates, then we may write $a = (a_1, a_2)$. According to this convention, we may also write $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

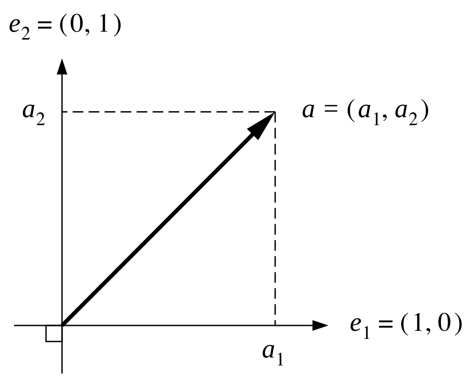

Figure 1. The coordinates of a vector a relative to two perpendicular axes

The directed line segment running from the origin 0 to the point a is described as a geometric vector which is bound to the origin. The ordered pair $(a_1, a_2) = a$ may be described as an algebraic vector. In fact, it serves little purpose to make a distinction between these two entities—the algebraic vector and the geometric vector—which may be regarded hereafter as alternative representations of the same object a. The unit vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$, which serve, in fact, to define the coordinate system, are described as the basis vectors.

The sum of two vectors $a = (a_1, a_2)$ and $b = (b_1, b_2)$ is defined by

$$\begin{aligned} a + b &= (a_1, a_2) + (b_1, b_2) \\ &= (a_1 + b_1, a_2 + b_2). \end{aligned} \tag{1}$$

The geometric representation of vector addition corresponds to a parallelogram of forces. Forces, which have both magnitude and direction, may be represented by directed line segments whose lengths correspond to the magnitudes. Hence forces may be described as vectors; and, as such, they obey the law of addition given above.

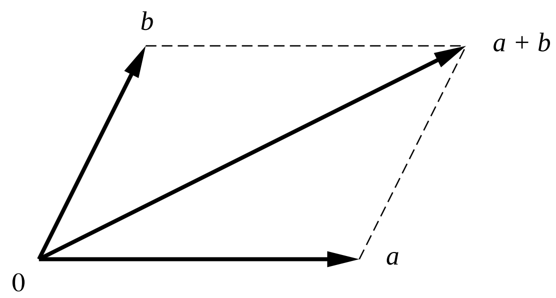

Figure 2. The parallelogram law of vector addition

If $a = (a_1, a_2)$ is a vector and λ is a real number, which is also described as a scalar, then the product of a and λ is defined by

$$\begin{aligned} \lambda a &= \lambda(a_1, a_2) \\ &= (\lambda a_1, \lambda a_2). \end{aligned} \tag{2}$$

The geometric counterpart of multiplication by a scalar is a stretching or a contraction of the vector which affects its length but not its direction.

The axes of the coordinate system are provided by the lines $\mathcal{E}_1 = \{\lambda e_1\}$ and $\mathcal{E}_2 = \{\lambda e_2\}$ which are defined by letting λ take every possible value. In terms of the basis vectors e_1 and e_2, the point $a = (a_1, a_2)$ can be represented by

$$\begin{aligned} a &= (a_1, a_2) \\ &= a_1 e_1 + a_2 e_2. \end{aligned} \tag{3}$$

Norms, inner products and angles

The length or norm of a vector $a = (a_1, a_2)$ is

$$\|a\| = \sqrt{a_1^2 + a_2^2}; \tag{4}$$

and this may be regarded either as an algebraic definition or as a consequence of the geometric theorem of Pythagoras.

The inner product of two vectors $a = (a_1, a_2)$ and $b = (b_1, b_2)$ may be defined by

$$\begin{aligned} \langle a, b\rangle &= \frac{1}{2}\Big\{\|a\|^2 + \|b\|^2 - \|a-b\|^2\Big\} \\ &= \frac{1}{2}\Big\{(a_1^2 + a_2^2) + (b_1^2 + b_2^2) - \big[(a_1 - b_1)^2 + (a_2 - b_2)^2\big]\Big\} \\ &= a_1 b_1 + a_2 b_2. \end{aligned} \tag{5}$$

For an alternative notation we may use

$$a'b = \langle a, b\rangle. \tag{6}$$

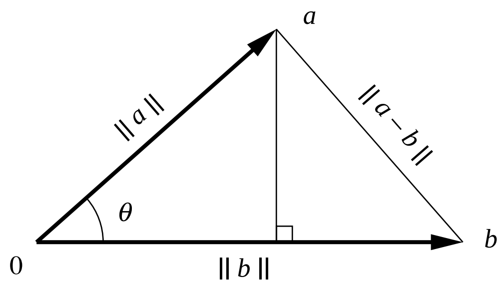

Figure 3. The law of cosines.

To provide an equivalent definition for the inner product, we may prove the law of cosines. From the diagram, by Pythagoras,

$$\begin{aligned} \|a-b\|^2 &= \|a\|^2 \sin^2\theta + \Big\{\|b\| - \|a\|\cos\theta\Big\}^2 \\ &= \|a\|^2 \sin^2\theta + \|b\|^2 + \|a\|^2\cos^2\theta - 2\|a\|.\|b\|\cos\theta \\ &= \|a\|^2 + \|b\|^2 - 2\|a\|.\|b\|\cos\theta. \end{aligned} \tag{7}$$

This enables us to rewrite our definition of the inner product of a and b as

$$\langle a, b\rangle = \|a\|.\|b\|\cos\theta, \tag{8}$$

where θ is the angle between the (geometric) vectors a and b.

If we were to take a strictly algebraic view, then this equation would serve not as a definition of the inner product, which is already defined in (5) above, but as a definition of the angle between the algebraic vectors a and b. Notice that, if θ has the value of a right angle, that is if $\theta = \pi/2$, then $\langle a, b\rangle = 0$. In that case the vectors a, b are said to be orthogonal or perpendicular to each other.

If p and a are two vectors and if θ is the angle between them, then the component of p in the direction of a is the scalar

$$c = \|p\| \cos\theta = \frac{\langle a, p\rangle}{\|a\|}. \tag{9}$$

From this definition, we see, in particular, that the components of a vector a in the direction of the basis vectors e_1 and e_2 are the Cartesian coordinates

$$a_1 = \langle e_1, a\rangle \quad \text{and} \quad a_2 = \langle e_2, a\rangle; \tag{10}$$

which follows when we use $\|e_1\| = \|e_2\| = 1$.

The (orthogonal) projection of p on a is the vector $\hat{p} = \lambda a$ of length $c = \|p\| \cos\theta$. Thus

$$\hat{p} = c\frac{a}{\|a\|} = a\frac{\langle a, p\rangle}{\langle a, a\rangle}. \tag{11}$$

Confusion can arise when, as is common, it is the length c of this vector which is described as the projection.

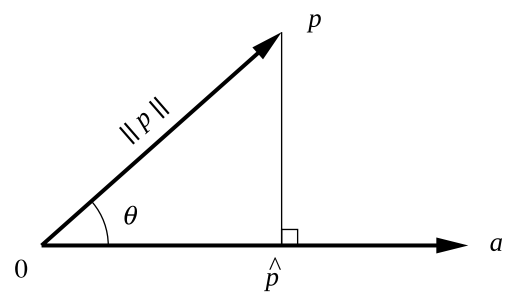

Figure 4. The perpendicular projection of p on a.

The equation of a line

Let $\mathcal{L}$ be a line from the origin 0 and let b be any vector (or point) lying along $\mathcal{L}$. Then, if x is any other vector in the line, there exists a scalar λ such that $x = \lambda b$.

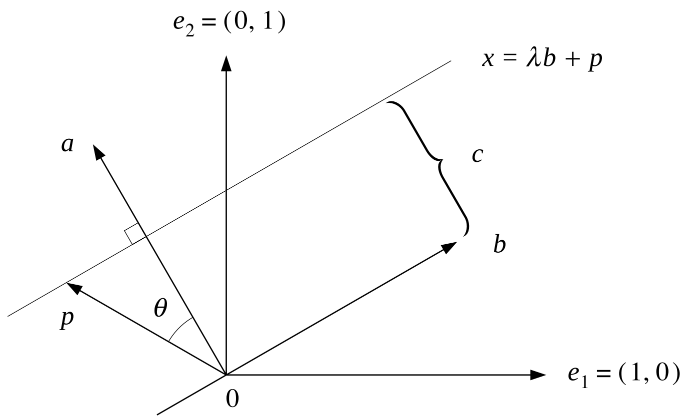

Figure 5. The line $x = \lambda b + p$ parallell to b and passing through the point p.

Now let $\mathcal{L}$ be a line parallel to b passing through a point p. Then if x is a point in $\mathcal{L}$ it can be expressed as

$$x = \lambda b + p, \tag{12}$$

for some scalar λ. This is called the parametric equation of the line. To derive an alternative expression, consider a vector a perpendicular to $\mathcal{L}$ such that $\langle a, b\rangle = a'b = 0$. On premultiplying the equation (12) by a', that is to say, by forming the inner product of a and x, we get

$$\begin{aligned} a'x &= \lambda a'b + a'p \\ &= a'p = c; \end{aligned} \tag{13}$$

and our equation for the line now has the form of

$$a_1x_1 + a_2x_2 = c, \tag{14}$$

where a_1, a_2 and c are constants.

Since a is subject only to the condition that $a'b = 0$, it is not uniquely determined—clearly, the choice of a also determines the value of the scalar c. Let us specify that a has unit length such that $\|a\| = \sqrt{a_1^2 + a_2^2} = 1$. Then it follows that

$$\begin{aligned} \langle a, p\rangle = a'p &= \|a\|.\|p\| \cos\theta \\ &= \|p\| \cos\theta = c; \end{aligned} \tag{15}$$

from which we see that c, which is the length of the projection of p on a, is the perpendicular distance of the line $\mathcal{L}$ from the origin 0.

When $\sqrt{a_1^2 + a_2^2} = 1$, we say that equation (14) is in *canonical form.* The equation is in *intercept form* when it is written as

$$a_1 x_1 + a_2 x_2 = 1. \tag{16}$$

In that case, a_1 is the reciprocal of the intercept of the line with the axis of e_1: for then we have

$$x_1 + \frac{a_2}{a_1} x_2 = \frac{1}{a_1}, \tag{17}$$

whence, with $x_2 = 0$, and we get $x_1 = 1/a_1$.

The distance of a point from a line

Let q be a point and let $a'x = c$, with $\|a\| = 1$, be the equation of the line $\mathcal{L}$. We wish to find the perpendicular distance from q to $\mathcal{L}$. This is also the minimum distance.

By the previous argument, the (perpendicular) distance from the origin to the line which passes through q and which is parallel to $\mathcal{L}$ is just $a'q = d$. But we already know that the distance from $\mathcal{L}$ to the origin is $a'x = c$. Therefore it follows that the distance from q to $\mathcal{L}$ is

$$d - c = a'q - c. \tag{18}$$

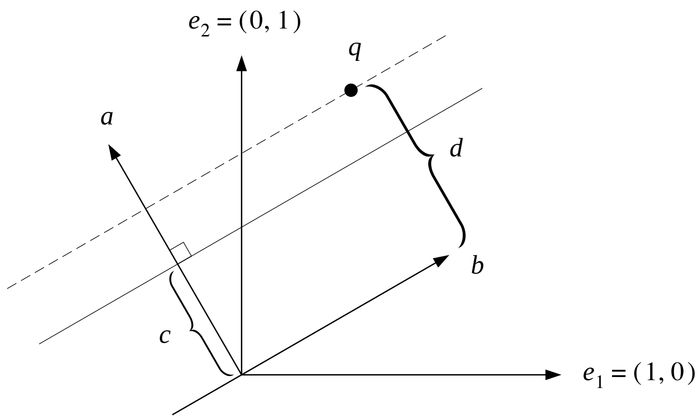

Figure 6. The distance of a point from a line.

Simultaneous equations

Consider the equations

$$\begin{aligned} ax + by &= e, \\ cx + dy &= f, \end{aligned} \tag{19}$$

which describe two lines in the plane. The coordinates (x, y) of the point of intersection of the lines is the algebraic solution of the simultaneous equations.

The equations may be written in matrix form as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}. \tag{20}$$

The necessary and sufficient condition for the existence of a unique solution is that

$$\text{Det} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \neq 0. \tag{21}$$

Then the solution is given by

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} e \\ f \end{bmatrix} \\ &= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix}. \end{aligned} \tag{22}$$

We may prove that

$$\begin{aligned} &\text{Det} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0 \quad \text{if and only if} \\ &a = \lambda c \quad \text{and} \quad b = \lambda d \quad \text{for some scalar} \quad \lambda. \end{aligned} \tag{23}$$

Proof. From $a = \lambda c$ and $b = \lambda d$ we derive, by cross multiplication, the identity $\lambda ad = \lambda bc$, whence $ad - bc = 0$ and the determinant is zero. Conversely, if $ad - bc = 0$, then we can deduce that $a = (b/d)c$ and $b = (a/c)d$ together with the identity $(b/d) = (a/c) = \lambda$, which implies that $a = \lambda c$ and $b = \lambda d$.

When the determinant is zero-valued one of two possibilities ensues. The first in when $e = \lambda f$. Then the two equations describe the same line and there is infinite number of solutions, with each solution corresponding to a point on the line. The second possibility is when $e \neq \lambda f$. Then the equations describe parallel lines and there are no solutions. Therefore, we say that the equations are inconsistent.

It is appropriate to end this section by giving a geometric interpretation of

$$\text{Det}\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = a_1b_2 - a_2b_1. \tag{24}$$

This is simply the area enclosed by the parallelogram of forces which is formed by adding the vectors $a = (a_1, a_2)$ and $b = (b_1, b_2)$. The result can be established by subtracting triangles from the rectangle in the accompanying figure to show that the area of the shaded region is $\frac{1}{2}(a_1b_2 - a_2b_1)$. The shaded region comprises half of the area which is enclosed by the parallelogram of forces.

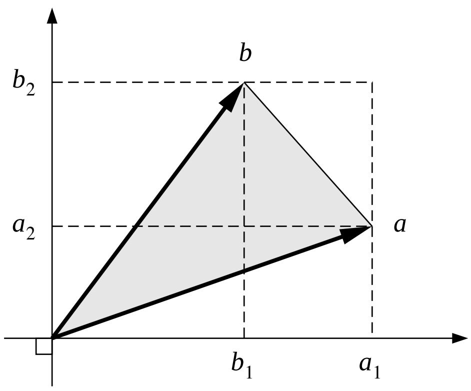

Figure 7. The determinant corresponds to the area enclosed by the parallelogram of forces.

The least-squares derivation of the E-V-M estimating equations

We have asserted that, in the case where $V(\eta_1) = V(\eta_2)$ and $C(\eta_1, \eta_2) = 0$, the estimating equations of the errors-in-variables model could be derived from the criterion of minimising the error sum of squares

$$\sum_{t=1}^{T}\Big\{(y_{1t} - \xi_{1t})^2 + (y_{2t} - \xi_{2t})^2\Big\} \tag{25}$$

subject to the constraint that

$$\xi_1\beta_1 + \xi_2\beta_2 = \alpha, \quad \text{where} \quad \beta_1^2 + \beta_2^2 = 1. \tag{26}$$

This amounts to minimising the sum of squares of the perpendicular distances of the points (y_1, y_2) from the regression line defined by equation (26).

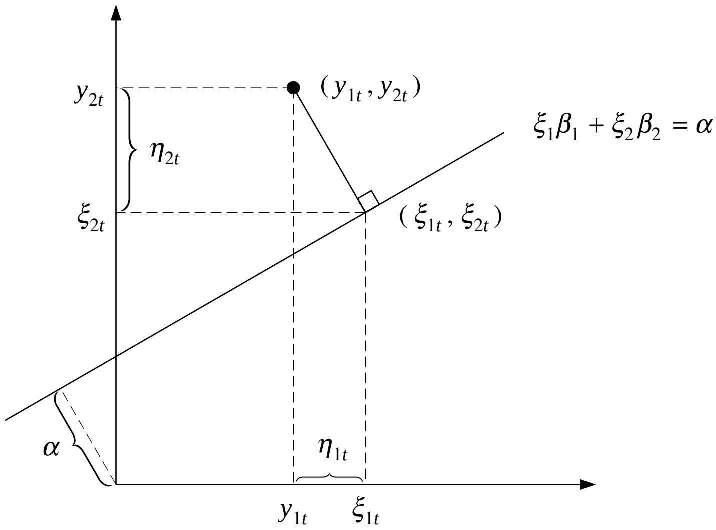

Figure 8. The perpendicular projection of the point (y_{1t}, y_{2t}) onto the regression line $\xi_1\beta_1 + \xi_2\beta_2 = \alpha$.

From previous results in this appendix, it follows that the minimand may be expressed as

$$\sum_{t=1}^{T} \big(\beta_1 y_{1t} + \beta_2 y_{2t} - \alpha\big)^2. \tag{27}$$

To evaluate this criterion, we may form the following lagrangean expression:

$$L = \sum_{t=1}^{T} \big(\beta_1 y_{1t} + \beta_2 y_{2t} - \alpha\big)^2 - \mu(\beta_1^2 + \beta_2^2 - 1). \tag{28}$$

Differentiating with respect to α and setting the result to zero for a minimum gives

$$\frac{\partial L}{\partial \alpha} = -2\sum \big(\beta_1 y_{1t} + \beta_2 y_{2t} - \alpha\big) = 0, \tag{29}$$

whence, on dividing by $2T$ and rearranging, we get the estimating equation for α:

$$\alpha(\beta_1, \beta_2) = \beta_1 \bar{y}_1 + \beta_2 \bar{y}_2. \tag{30}$$

By substituting the latter into the criterion function, we may form a concentrated function which has β_1 and β_2 as its sole arguments:

$$L^c = \sum_{t=1}^{T} \bigl\{\beta_1(y_{1t} - \bar{y}_1) + \beta_2(y_{2t} - \bar{y}_2)\bigr\}^2 - \mu(\beta_1^2 + \beta_2^2 - 1). \tag{31}$$

The first-order conditions for the minimum are

$$\begin{aligned} \frac{\partial L^c}{\partial \beta_1} &= 2\sum \beta_1(y_{1t} - \bar{y}_1)^2 + 2\sum \beta_2(y_{1t} - \bar{y}_1)(y_{2t} - \bar{y}_2) - 2\mu\beta_1 = 0, \\ \frac{\partial L^c}{\partial \beta_2} &= 2\sum \beta_1(y_{2t} - \bar{y}_2)(y_{1t} - \bar{y}_1) + 2\sum \beta_2(y_{2t} - \bar{y}_2)^2 - 2\mu\beta_2 = 0. \end{aligned} \tag{32}$$

Having divided both equations throughout by $2T$, we may assemble them in the following matrix format

$$\left\{ \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{33}$$

wherein s_{11}, s_{12} and s_{22} are the moments defined in (18). Equation (33) is precisely the estimating equation which has been given under (2.25). Apart from the proliferation of symbols, there should be no difficulty in deriving the estimating equation under (2.24) from the generalised least-squares criterion function under (2.28).