Characteristic Functions and the Uncertainty Relationship

There is an inverse relationship between the dispersion of a function and the range of the frequencies which are present in its transform. Thus one finds that, the shorter is the duration of a transient signal, the wider is the spread of the frequencies in its transform.

In electrical engineering, this notion finds expression in the so-called bandwidth theorem. In mathematical physics, an analogous relationship between the spatial dispersion of a wave train and its frequency dispersion is the basis of the uncertainty principle of Heisenberg.

To illustrate the relationship, we may consider a Gaussian or normal distribution. This is defined in terms of the random variable x by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2\sigma^2}(x-\mu)^2 \right\}. \tag{88}$$

The Fourier transform of $f(x)$, which in known in mathematical statistics as the characteristic function of the normal distribution, is given by

$$\begin{aligned} \phi(\omega) &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left\{ i\omega x - \frac{1}{2\sigma^2}(x-\mu)^2 \right\} dx \\ &= \exp\left\{ i\omega\mu - \frac{1}{2}\sigma^2\omega^2 \right\} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left\{ -\frac{1}{2\sigma^2}(x-\mu-i\sigma^2\omega)^2 \right\} dx. \end{aligned} \tag{89}$$

The integral here is that of the function $\exp\{z^2/(2\sigma^2)\}$, where z is a complex variable which runs along a line parallel to the real axis. This can be show to be equal to integral of the corresponding real function which has the value of $\sigma\sqrt{2\pi}$. Therefore the characteristic function is

$$\phi(\omega) = \exp\left\{ i\omega\mu - \frac{1}{2}\sigma^2\omega^2 \right\}. \tag{90}$$

The characteristic function is so-called because it completely characterises the distribution. The parameters of the distribution are the mean μ and the variance σ^2 which measures the dispersion of x. The distribution is symmetric about the value μ; and if, $\mu = 0$, then $\phi(\omega)$ is real-valued, as we are led to expect from the symmetry properties of the Fourier transform.

The inverse relationship between the dispersions of $f(x)$ and $\phi(\omega)$ is manifest from the comparison of the two functions which, apart from a scalar factor, have the same form when $\mu = 0$. Thus, if the dispersion of $f(x)$ is represented by σ, then that of $\phi(\omega)$ is directly related to σ^{-1}.

The measure of dispersion which is used in mathematical statistics, and which is based on the presumption that the function is nonnegative, is inappropriate for measuring the width of an oscillatory signal or a wave train. In such cases, the usual measure of dispersion of x is

$$\Delta_x^2 = \int_{-\infty}^{\infty} x^2 |f(x)|^2 dx \bigg/ \int_{-\infty}^{\infty} |f(x)|^2 dx. \tag{91}$$

The dispersion Δ_ω^2 in the frequency domain is defined likewise.

In quantum mechanics, a particle is also depicted as a De Broglie wave. Schrödinger's wave equation $\psi(x)$ serves to define the spatial extent of the wave train, and its dispersion Δ_x is liable to be interpreted as a measure of the uncertainty of our knowledge of the particle's position.

The formulation of De Broglie also relates the momentum ρ of a particle to its wavelength $\lambda = 1/\omega$ according to the formula $\rho = h/\lambda$, where h is Plank's constant. Thus the spread of momentum is $\Delta_\rho = h\Delta_\omega$; and the position–momentum uncertainty principle states that

$$\Delta_x \Delta_\rho \geq \frac{h}{4\pi}. \tag{92}$$

It can be shown that the Gaussian wave train is the only one which leads to an equality in this relationship.