The Moment Generating Function of the Normal Distribution

Recall that the probability density function of a normally distributed random variable x with a mean of $E(x) = \mu$ and a variance of $V(x) = \sigma^2$ is

$$N(x; \mu, \sigma^2) = \frac{1}{\sqrt{(2\pi\sigma^2)}} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}. \tag{1}$$

Our object is to find the moment generating function which corresponds to this distribution. To begin, let us consider the case where $\mu = 0$ and $\sigma^2 = 1$. Then we have a standard normal, denoted by $N(z; 0, 1)$, and the corresponding moment generating function is defined by

$$\begin{aligned} M_z(t) = E(e^{zt}) &= \int e^{zt} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\ &= e^{\frac{1}{2}t^2}. \end{aligned} \tag{2}$$

To demonstate this result, the exponential terms may be gathered and rearranged to give

$$\begin{aligned} \exp\{zt\} \exp\{-\tfrac{1}{2}z^2\} &= \exp\{-\tfrac{1}{2}z^2 + zt\} \\ &= \exp\{-\tfrac{1}{2}(z-t)^2\} \exp\{\tfrac{1}{2}t^2\}. \end{aligned} \tag{3}$$

Then

$$\begin{aligned} M_z(t) &= e^{\frac{1}{2}t^2} \int \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-t)^2} dz \\ &= e^{\frac{1}{2}t^2}, \end{aligned} \tag{4}$$

where the final equality follows from the fact that the expression under the integral is the $N(z; \mu = t, \sigma^2 = 1)$ probability density function which integrates to unity.

Now consider the moment generating function of the Normal $N(x; \mu, \sigma^2)$ distribution when μ and σ^2 have arbitrary values. This is given by

$$M_x(t) = E(e^{xt}) = \int e^{xt} \frac{1}{\sqrt{(2\pi\sigma^2)}} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2} dx \tag{5}$$

Define

$$z = \frac{x - \mu}{\sigma}, \quad \text{which implies} \quad x = z\sigma + \mu. \tag{6}$$

Then, using the change-of-variable technique, we get

$$
\begin{aligned}
M_x(t) &= e^{\mu t}\int e^{z\sigma t}\frac{1}{\sqrt{(2\pi\sigma^2)}}e^{-\frac{1}{2}z^2}\left|\frac{dx}{dz}\right|dz \\
&= e^{\mu t}\int e^{z\sigma t}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz \qquad (7)\\
&= e^{\mu t}e^{\frac{1}{2}\sigma^2 t^2},
\end{aligned}
$$

Here, to establish the first equality, we have used $dx/dz = \sigma$. The final equality follows in view of the final equality of equation (2) above where the dummy variable t can be replace by σt. The conclusion is that

(8) The moment generating function corresponding to the normal probability density function $N(x;\mu,\sigma^2)$ is the function $M_x(t) = \exp\{\mu t + \sigma^2 t^2/2\}$.

The notable characteristic of this function is that it is in the form of an exponential. This immediately implies that the sum of two independently distributed Normal random variables is itself a normally distributed random variable. Thus:

(9) Let $x \sim N(\mu_x, \sigma_x^2)$ and $y \sim N(\mu_y, \sigma_y^2)$ be two independently distributed normal variables. Then their sum is also a normally distributed random variable: $x + y = z \sim N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$.

To prove this we need only invoke the result that, in the case of independence, the moment generating function of the sum is the product of the moment generating functions of its elements. This enables us to write

$$
\begin{aligned}
M_z(t) &= M_x(t)M_y(t) = \exp\left\{\mu_x t + \tfrac{1}{2}\sigma_x^2 t^2\right\}\exp\left\{\mu_y t + \tfrac{1}{2}\sigma_y^2 t^2\right\} \qquad (10)\\
&= \exp\left\{(\mu_x + \mu_y)t + \tfrac{1}{2}(\sigma_x^2 + \sigma_y^2)t^2\right\}
\end{aligned}
$$

which is recognised as the moment generating function of a normal distribution.