SETS AND SUBSETS

Definitions

(1) A *set* A in any collection of objects which have a common characteristic. If an object x has the characteristic, then we say that it is an *element* or a *member* of the set and we write $x \in A$. If x is not a member of the set A, then we write $x \notin A$.

It may be possible to specify a set by writing down all of its elements. If x, y, z are the only elements of the set A, then the set can be written as

(2) $$A = \{x, y, z\}.$$

Similarly, if A is a finite set comprising n elements, then it can be denoted by

(3) $$A = \{x_1, x_2, \ldots, x_n\}.$$

The three dots, which stand for the phase "and so on", are called an ellipsis. The subscripts which are applied to the x's place them in a one-to-one correspondence with set of integers $\{1, 2, \ldots, n\}$ which constitutes the so-called index set. However, this indexing imposes no necessary order on the elements of the set; and its only pupose is to make it easier to reference the elements.

Sometimes we write an expression in the form of

(4) $$A = \{x_1, x_2, \ldots\}.$$

Here the implication is that the set A may have an infinite number of elements; in which case a correspondence is indicated between the elements of the set and the set of *natural numbers* or *positive integers* which we shall denote by

(5) $$\mathcal{Z} = \{1, 2, \ldots\}.$$

(Here the letter $\mathcal{Z}$, which denotes the set of natural numbers, derives from the German verb *zahlen:* to count)

An alternative way of denoting a set is to specify the characteristic which is common to its elements. Examples are provided by

(6) $$A = \{x;\ x \text{ is a cracked billiard ball}\},$$

and by

(7) $$A = \left\{x;\ x = \tfrac{1}{n}, n \in \mathcal{Z}\right\}.$$

The latter is just another way of denoting the set

$$A = \{1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots\}. \tag{8}$$

It may seem like an abstruse way; but, in fact, it has the advantage of being unambiguous.

Sometimes, we are bound to use such notation for the reason that the elements of the set cannot be *enumerated.* That is to say, it may not be possible to establish a correspondence between the elements of the set and a set of consecutive integers. An example is provided by the set of all positive real numbers which can be denoted by

$$\mathcal{R}^+ = \{x;\ x \geq 0\}. \tag{9}$$

We shall be somewhat more specific about the contents of this set at a later stage.

One should make a clear distinction between the set $A = \{x\}$ and the the singleton element x. Thus it makes sense to write $x \in \{x\} = A$; but it is incorrect to assert that $A \in A$, since a set is not the same as any or all of its elements.

It is helpful to have a notation for the special set which contains no elements. This might be denoted simply by writing an empty pair of braces $\{\}$. In fact, the symbol $\emptyset$ is commonly used. By definition, a statement such as $x \in \emptyset$ must be false, since the empty set contains no elements. It is certainly true that the empty set $\emptyset$ and the number zero 0 are two quite different entities. The notation for the empty set, which may have originated as a zero struck through, is a helpful reminder of this point.

A basic relationship between sets is that of inclusion which entails the definition of a subset:

(10) We say that B is a *subset* of A, and we write $B \subset A$ or $A \supset B$ (i.e. A includes B) if every element of B is also an element of A. Alternatively, we may express this relationship by asserting that, if $x \in B$, then $x \in A$, or simply by asserting that $x \in B$ implies $x \in A$.

Observe that, if $A \subset B$ and $B \subset A$, then A and B are one and the same; and we may write $A = B$. If $B \subset A$ and $B \neq A$, then B is a *proper subset* of A.

In any context, there is a set which contain all others. This set is called the *universal set,* denoted by $\mathcal{S}$.

Operations on Sets

(11) A binary operation of *union,* denoted by the symbol $\cup$, may be defined relative to any two sets A and B. The operation generates the set

$$A \cup B = \{x;\ x \in A \text{ or } x \in B\}.$$

Here the word "or" is used in the inclusive sense to imply that x is either in A or in B or in both. For example, if S is the set of all vertebrates, A is the characteristic of having fur and B is the characteristic of laying eggs, then $A \cup B$ certainly has the duck-bill platypus amongst its elements as well as foxes and geese.

(12) A binary operation of *intersection,* denoted by the symbol $\cap$, may be defined relative to any two sets A and B. The operation generates the set

$$A \cap B = \{x;\ x \in A \text{ and } x \in B\}.$$

In terms of the previous example, $A \cap B$ (unless I am mistaken) has only the duck-bill platypus and the spiny ant eater as its two elements.

(13) Two sets A and B are said to be *disjoint* if their intersection is the empty set

$$A \cap B = \emptyset.$$

If A is the set of vertebrate fish and B is the set of mammals, then, according to modern usage, their intersection is the empty set. However, as recently as Victorian times, whales, which are mammals, were liable to be described as fish.

(14) Let $A \subset S$. Then the *complement* of A in S, denoted by A^c, is the set of all the elements of S which do not belong to A:

$$A^c = \{x;\ x \notin A\}.$$

The Rules of Boolean Algebra

The binary operations of union $\cup$ and intersection $\cap$ are roughly analogous to the arithmetic operations of addition $+$ and multiplication $\times$, and they obey a similar set of laws. In fact, the laws of Boolean algebra are virtually symmetric with respect to the two operations in the sense that, in any statement of the laws, the symbols can be interchanged without affecting the truth of the statements. This in not the case in arithmetic. The laws are as follows:

(15) Commutative law:
$$A \cup B = B \cup A,$$
$$A \cap B = B \cap A,$$

(16) Associative law:
$$(A \cup B) \cup C = A \cup (B \cup C),$$
$$(A \cap B) \cap C = A \cap (B \cap C),$$

(17) Distributive law:
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$

$$\text{(18)} \qquad \text{Idempotency law:} \quad A \cup A = A, \quad A \cap A = A.$$

These various laws have the status of axioms.

There are several useful identities which are deducible from the axioms. Thus *De Morgan's Rules* state that

$$\text{(19)} \qquad (A \cup B)^c = A^c \cap B^c \qquad \text{and} \qquad (A \cap B)^c = A^c \cup B^c.$$

Amongst other useful results are the following:

$$\text{(20)} \qquad \begin{array}{rlrl} \text{(i)} & A \cup A^c = S, & \text{(iv)} & A \cap S = A, \\ \text{(ii)} & A \cap A^c = \emptyset, & \text{(v)} & A \cup \emptyset = A, \\ \text{(iii)} & A \cup S = S, & \text{(vi)} & A \cap \emptyset = \emptyset. \end{array}$$

Problems

1. Prove De Morgans' rules using a Venn diagram.

2. Substitute the symbols $+$ for addition and $\times$ for multiplication in place of $\cup$ and $\cap$ respectively in the statements (15)–(18) of the laws of Boolean algebra. Determine whether the resulting statements concerning the arithmetic operations are true or false. Attempt to give a complete statement of the rules of arithmetic.

3. Evaluate the following expressions:

(a) $A \cup (B^c \cup A)^c$,

(b) $A \cap (B \cup A^c)$,

(c) $A \cap (B^c \cap A)^c$.