

### Cochrane's Theorem: The Decomposition of a Chi-Square Variate

The standard test of an hypothesis regarding the vector  $\beta$  in the model  $N(y; X\beta, \sigma^2 I)$  entails a multi-dimensional version of Pythagoras' Theorem. Consider the decomposition of the vector  $y$  into the systematic component and the residual vector. This gives

$$(1) \quad \begin{aligned} y &= X\hat{\beta} + (y - X\hat{\beta}) \quad \text{and} \\ y - X\beta &= (X\hat{\beta} - X\beta) + (y - X\hat{\beta}), \end{aligned}$$

where the second equation comes from subtracting the unknown mean vector  $X\beta$  from both sides of the first. These equations can also be expressed in terms of the projector  $P = X(X'X)^{-1}X'$  which gives  $Py = X\hat{\beta}$  and  $(I - P)y = y - X\hat{\beta} = e$ . Using the definition  $\varepsilon = y - X\beta$  within the second of the equations, we have

$$(2) \quad \begin{aligned} y &= Py + (I - P)y \quad \text{and} \\ \varepsilon &= P\varepsilon + (I - P)\varepsilon. \end{aligned}$$

The reason for rendering the equation in this notation is that it enables us to envisage more clearly the Pythagorean relationship between the vectors. Thus, using the fact that  $P = P' = P^2$  and the fact that  $P'(I - P) = 0$ , it can be established that

$$(3) \quad \begin{aligned} \varepsilon'\varepsilon &= \varepsilon'P\varepsilon + \varepsilon'(I - P)\varepsilon \quad \text{or} \\ \varepsilon'\varepsilon &= (X\hat{\beta} - X\beta)'(X\hat{\beta} - X\beta) + (y - X\hat{\beta})'(y - X\hat{\beta}). \end{aligned}$$

The terms in these expressions represent squared lengths; and the vectors themselves form the sides of a right-angled triangle with  $P\varepsilon$  at the base,  $(I - P)\varepsilon$  as the vertical side and  $\varepsilon$  as the hypotenuse.

The usual test of an hypothesis regarding the elements of the vector  $\beta$  is based on the foregoing relationships. Imagine that the hypothesis postulates that the true value of the parameter vector is  $\beta_0$ . To test this notion, we compare the value of  $X\beta_0$  with the estimated mean vector  $X\hat{\beta}$ . The test is a matter of assessing the proximity of the two vectors which is measured by the square of the distance which separates them. This would be given by  $\varepsilon'P\varepsilon = (X\hat{\beta} - X\beta_0)'(X\hat{\beta} - X\beta_0)$  if the hypothesis were true. If the hypothesis is untrue and if  $X\beta_0$  is remote from the true value of  $X\beta$ , then the distance is liable to be excessive. The distance can only be assessed in comparison with the variance  $\sigma^2$  of the disturbance term or with an estimate thereof. Usually,

one has to make do with the estimate of  $\sigma^2$  which is provided by

$$(4) \quad \begin{aligned} \hat{\sigma}^2 &= \frac{(y - X\hat{\beta})'(y - X\hat{\beta})}{T - k} \\ &= \frac{\varepsilon'(I - P)\varepsilon}{T - k}. \end{aligned}$$

The numerator of this estimate is simply the squared length of the vector  $e = (I - P)y = (I - P)\varepsilon$  which constitutes the vertical side of the right-angled triangle.

The test uses the result that

$$(5) \quad \text{If } y \sim N(X\beta, \sigma^2 I) \text{ and if } \hat{\beta} = (X'X)^{-1}X'y, \text{ then}$$

$$F = \left\{ \frac{(X\hat{\beta} - X\beta)'(X\hat{\beta} - X\beta)}{k} \middle/ \frac{(y - X\hat{\beta})'(y - X\hat{\beta})}{T - k} \right\}$$

is distributed as an  $F(k, T - k)$  statistic.

This result depends upon Cochran's Theorem concerning the decomposition of a chi-square random variate. The following is a statement of the theorem which is attuned to our present requirements:

$$(6) \quad \text{Let } \varepsilon \sim N(0, \sigma^2 I_T) \text{ be a random vector of } T \text{ independently and identically distributed elements. Also let } P = X(X'X)^{-1}X' \text{ be a symmetric idempotent matrix, such that } P = P' = P^2, \text{ which is constructed from a matrix } X \text{ of order } T \times k \text{ with } \text{Rank}(X) = k. \text{ Then}$$

$$\frac{\varepsilon'P\varepsilon}{\sigma^2} + \frac{\varepsilon'(I - P)\varepsilon}{\sigma^2} = \frac{\varepsilon'\varepsilon}{\sigma^2} \sim \chi^2(T),$$

which is a chi-square variate of  $T$  degrees of freedom, represents the sum of two independent chi-square variates  $\varepsilon'P\varepsilon/\sigma^2 \sim \chi^2(k)$  and  $\varepsilon'(I - P)\varepsilon/\sigma^2 \sim \chi^2(T - k)$  of  $k$  and  $T - k$  degrees of freedom respectively.

To prove this result, we begin by finding an alternative expression for the projector  $P = X(X'X)^{-1}X'$ . First consider the fact that  $X'X$  is a symmetric positive-definite matrix. It follows that there exists a matrix transformation  $T$  such that  $T(X'X)T' = I$  and  $T'T = (X'X)^{-1}$ . Therefore  $P = XT'TX' = C_1C_1'$ , where  $C_1 = XT'$  is a  $T \times k$  matrix comprising  $k$  orthonormal vectors such that  $C_1'C_1 = I_k$  is the identity matrix of order  $k$ .

Now define  $C_2$  to be a complementary matrix of  $T - k$  orthonormal vectors. Then  $C = [C_1, C_2]$  is an orthonormal matrix of order  $T$  such that

$$(7) \quad \begin{aligned} CC' &= C_1C_1' + C_2C_2' = I_T \quad \text{and} \\ C'C &= \begin{bmatrix} C_1'C_1 & C_1'C_2 \\ C_2'C_1 & C_2'C_2 \end{bmatrix} = \begin{bmatrix} I_k & 0 \\ 0 & I_{T-k} \end{bmatrix}. \end{aligned}$$

The first of these results allows us to set  $I - P = I - C_1C_1' = C_2C_2'$ . Now, if  $\varepsilon \sim N(0, \sigma^2 I_T)$  and if  $C$  is an orthonormal matrix such that  $C'C = I_T$ , then it follows that  $C'\varepsilon \sim N(0, \sigma^2 I_T)$ . In effect, if  $\varepsilon$  is a normally distributed random vector with a density function which is centred on zero and which has spherical contours, and if  $C$  is the matrix of a rotation, then nothing is altered by applying the rotation to the random vector. On partitioning  $C'\varepsilon$ , we find that

$$(8) \quad \begin{bmatrix} C_1'\varepsilon \\ C_2'\varepsilon \end{bmatrix} \sim N \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^2 I_k & 0 \\ 0 & \sigma^2 I_{T-k} \end{bmatrix} \right),$$

which is to say that  $C_1'\varepsilon \sim N(0, \sigma^2 I_k)$  and  $C_2'\varepsilon \sim N(0, \sigma^2 I_{T-k})$  are independently distributed normal vectors. It follows that

$$(9) \quad \begin{aligned} \frac{\varepsilon' C_1 C_1' \varepsilon}{\sigma^2} &= \frac{\varepsilon' P \varepsilon}{\sigma^2} \sim \chi^2(k) \quad \text{and} \\ \frac{\varepsilon' C_2 C_2' \varepsilon}{\sigma^2} &= \frac{\varepsilon' (I - P) \varepsilon}{\sigma^2} \sim \chi^2(T - k) \end{aligned}$$

are independent chi-square variates. Since  $C_1 C_1' + C_2 C_2' = I_T$ , the sum of these two variates is

$$(10) \quad \frac{\varepsilon' C_1 C_1' \varepsilon}{\sigma^2} + \frac{\varepsilon' C_2 C_2' \varepsilon}{\sigma^2} = \frac{\varepsilon' \varepsilon}{\sigma^2} \sim \chi^2(T);$$

and thus the theorem is proved.

The statistic under (5) can now be expressed in the form of

$$(11) \quad F = \left\{ \frac{\varepsilon' P \varepsilon}{k} \bigg/ \frac{\varepsilon' (I - P) \varepsilon}{T - k} \right\}.$$

This is manifestly the ratio of two chi-square variates divided by their respective degrees of freedom; and so it has an  $F$  distribution with these degrees of freedom. This result provides the means for testing the hypothesis concerning the parameter vector  $\beta$ .