DISTRIBUTION THEORY

The Gamma Distribution

Consider the function

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx. \tag{1}$$

We shall attempt to integrate this. First recall that

$$\int u \frac{dv}{dx} = uv - \int v \frac{du}{dx} dx. \tag{2}$$

This method of integration is called integration by parts and it can be seen as a consequence of the product rule of differentiation. Let $u = x^{n-1}$ and $dv/dx = e^{-x}$. Then $v = -e^{-x}$ and

$$\begin{aligned}\int_0^\infty e^{-x} x^{n-1} dx &= \left[-x^{n-1} e^{-x} + \int e^{-x} (n-1) x^{n-2} dx \right]_0^\infty \\ &= (n-1) \int_0^\infty e^{-x} x^{n-2} dx.\end{aligned} \tag{3}$$

We may express the above by writing $\Gamma(n) = (n-1)\Gamma(n-1)$, from which it follows that

$$\Gamma(n) = (n-1)(n-2)\cdots\Gamma(\delta), \tag{4}$$

where $0 < \delta < 1$. Examples of the gamma function are

$$\Gamma(1/2) = \sqrt{\pi}, \tag{5}$$

$$\Gamma(1) = \int_0^\infty e^{-x} dx = \Big[-e^{-x} \Big]_0^\infty = 1, \tag{6}$$

$$\Gamma(n) = (n-1)(n-2)\cdots\Gamma(1) = (n-1)!. \tag{7}$$

Here it is assumed that n is an integer. The first of these results can be verified by confirming the following identities:

$$\begin{aligned}\sqrt{2\pi} &= \int_{-\infty}^\infty e^{-z^2/2} dz \\ &= \frac{1}{2}\sqrt{2} \int_{-\infty}^\infty e^{-x} x^{-1/2} dx = \sqrt{2}\Gamma(1/2).\end{aligned} \tag{8}$$

The first equality is familiar from the integration of the standard normal density function. The second equality follows when the variable of integration is changed from z to $x = z^2/2$, and the final equality invokes the definition of the gamma function, which is provided by equation (1).

Using the gamma function, we may define a probability density function known as the Gamma Type 1:

$$\gamma_1(x) = \frac{e^{-x}x^{n-1}}{\Gamma(n)} \qquad 0 < x < \infty. \tag{9}$$

For an integer value of n, the Gamma Type 1 gives the probability distribution of the waiting time to the nth event in a Poisson arrival process of unit mean. When $n = 1$, it becomes the exponential distribution, which relates to the waiting time for the first event.

To define the type 2 gamma function, we consider the transformation $z = \beta x$. Then, by the change-of-variable technique, we have

$$\begin{aligned} \gamma_2(z) &= \gamma_1\{x(z)\}\left|\frac{dx}{dz}\right| \\ &= \frac{e^{-z/\beta}(z/\beta)^{\alpha-1}}{\Gamma(\alpha)}\frac{1}{\beta}. \end{aligned} \tag{10}$$

Here we have changed the notation by setting $\alpha = n$. The function is written more conveniently as

$$\gamma_2(z; \alpha, \beta) = \frac{e^{-z/\beta}z^{\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}}. \tag{11}$$

Consider the γ_2 distribution where $\alpha = r/2$ with $r \in \{0, 1.2. \ldots\}$ and $\beta = 2$. This is the so-called chi-square distribution of r degrees of freedom:

$$\chi^2(r) = \frac{e^{-x/2}x^{(r/2)-1}}{\Gamma(r/2)2^{r/2}}. \tag{12}$$

Now let $w \sim N(0, 1)$ and consider $w^2 = y \sim g(y)$. Then $A = \{-\theta < w < \theta\} = \{0 < y < \theta^2\}$ defines an event which has the probability $P(A) = P\{0 < y < \theta^2\} = 2P\{0 < w < \theta\}$. Hence

$$\begin{aligned} P(A) &= 2\int_0^{\theta} N(w)dw = 2\int_0^{\theta^2} f\{w(y)\}\frac{dw}{dy}dy \\ &= 2\int_0^{\theta^2} \frac{1}{\sqrt(2\pi)}e^{-w^2/2}dw = 2\int_0^{\theta^2} \frac{1}{\sqrt(2\pi)}e^{-y/2}\frac{1}{2}y^{-1/2}dy. \end{aligned} \tag{13}$$

Under the integral of the final expression we have

$$f(y) = \frac{e^{-y/2}y^{1/2}}{\sqrt{\pi}\sqrt{2}} = \frac{e^{-y/2}y^{1/2}}{\Gamma(1/2)2^{1/2}} = \chi^2(2). \tag{14}$$

Hence $y \sim \chi^2(2)$.

The Moment Generating Function of the Gamma Distribution

Now let us endeavour to find the moment generating function of the γ_1 distribution. We have

$$M_x(t) = \int e^{xt} \frac{e^{-x}x^{n-1}}{\Gamma(n)} dx$$
(15)
$$= \int \frac{e^{-x(1-t)}x^{n-1}}{\Gamma(n)} dx.$$

Now let $w = x(1-t)$. Then, by the change-of-variable technique,

$$M_x(t) = \int \frac{e^{-w}w^{n-1}}{(1-t)^{n-1}\Gamma(n)} \frac{1}{(1-t)} dw$$
(16)
$$= \frac{1}{(1-t)^n} \int \frac{e^{-w}w^{n-1}}{\Gamma(n)} dw$$
$$= \frac{1}{(1-t)^n}.$$

We have defined the γ_2 distribution by

(17)
$$\gamma_2 = \frac{e^{-x/\beta}x^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha}; \qquad 0 \leq x < \infty.$$

Hence the moment generating function is defined by

$$M_x(t) = \int_0^\infty \frac{e^{tx}e^{-x/\beta}x^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} dx$$
(18)
$$= \int_0^\infty \frac{e^{-x(1-\beta t)/\beta}x^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} dx.$$

Let $y = x(1-\beta t)/\beta$. which gives $dy/dx = (1-\beta t)/\beta$. Then, by the change of variable technique we get

$$M_x(t) = \int_0^\infty \frac{e^{-y}}{\Gamma(\alpha)\beta^\alpha} \left(\frac{\beta y}{1-\beta t}\right)^{\alpha-1} \frac{\beta}{(1-\beta t)} dy$$
(19)
$$= \frac{1}{(1-\beta t)^\alpha} \int \frac{y^{\alpha-1}e^{-y}}{\Gamma(\alpha)} dt$$
$$= \frac{1}{(1-\beta t)^\alpha}.$$

SAMPLING THEORY

Our methods of statistical inference depend upon the procedure of drawing random samples from the population whose properties we wish to assess. We may regard the random sample as a microcosm of the population; and our inferential procedures can be described loosely as the process of assessing the properties of the samples and attributing them to the parent population. The procedures may be invalidated, or their usefulness may be prejudiced, on two accounts. First, any particular sample may have properties which diverge from those of the population in consequence of random error. Secondly, the process of sampling may induce systematic errors which cause the properties of the samples on average to diverge from those of the populations. We shall attempt to eliminate such systematic bias by adopting appropriate methods of inference. We shall also endeavour to assess the extent to which random errors weaken the validity of the inferences.

(20) A random sample is a set of n random variables $x_1, x_2, \ldots, x_n$ which are distributed independently according to a common probability density function. Thus $x_i \sim f(x_i)$ for all i, and the joint probability density function of the sample elements is $f(x_1, x_2, \ldots, x_n) = f(x_1)f(x_2)\cdots f(x_n)$.

Consider for example, the sample mean $\bar{x} = (x_1 + x_2 + \cdots + x_n)/n$ We have

$$E(\bar{x}) = E\left\{\frac{1}{n}\sum x_i\right\} = \frac{1}{n}\sum_{i=1}^{n} E(x_i) = \mu. \tag{21}$$

The variance of a sum of random variables is given generally by the formula

$$V\left(\sum x_i\right) = \sum_i V(x_i) + \sum_i\sum_j C(x_i, x_j), \tag{22}$$

where $i \neq j$. But, the independence of the variables implies that $C(x_i, x_j) = 0$ for all i, j; and, therefore,

$$V(\bar{x}) = \frac{1}{n^2}V\left(\sum x_i\right) = \frac{1}{n^2}\sum V(x_i) = \frac{\sigma^2}{n}. \tag{23}$$

Now consider the sample variance defined as

$$s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2. \tag{24}$$

The may be expanded as follows:

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 &= \frac{1}{n}\sum_{i=1}^{n}\{(x_i - \mu) - (\bar{x} - \mu)\}^2 \\ &= \frac{1}{n}\sum(x_i - \mu)^2 - \frac{2}{n}\sum(x_i - \mu)(\bar{x} - \mu) + (\bar{x} - \mu)^2 \\ &= \frac{1}{n}\sum(x_i - \mu)^2 - (\bar{x} - \mu)^2 \end{aligned} \tag{25}$$

It follows that

$$(26)\qquad \begin{aligned} E(s^2) &= \frac{1}{n}\sum E(x_i-\mu)^2 - E\{(\bar{x}-\mu)^2\} \\ &= \frac{1}{n}\{nV(x)+V(\bar{x})\} \\ &= \sigma^2 - \frac{\sigma^2}{n} = \sigma^2\frac{(n-1)}{n}. \end{aligned}$$

We conclude that the sample variance is a biased estimator of the population variance. To obtain an unbiased estimator, we use

$$(27)\qquad \hat{\sigma}^2 = \frac{1}{n-1}\sum(x_i-\bar{x})^2.$$

The Sampling Distributions

The processes of inference are enhanced if we can find the distributions of the various sampling statistics which interest us; for these will give us a better idea of the sampling error which besets the inferences. The expectations of the sample moments have already provided use with some of the parameters of these sampling distributions. We must now attempt to derive their functional forms from the probability distribution s of the parent populations. The following theorem is useful in this connection:

(28) Let $x = \{x_1, x_2, \ldots, x_n\}$ be a random vector with a probability density function $f(x)$ and let $u(x) \sim g(x)$ be a scalar-valued function of this vector. Then the moment generating function of $g(u)$ is

$$M\{u,t\} = E(e^{ut}) = \int_x e^{u(x)t} f(x)dx.$$

This result may enable us to find the moment generating function of a statistic which is a function of the vector of sample points $x = \{x_1, x_2, \ldots, x_n\}$. We might then recognise this moment-generating function as one which pertains to a know distribution, which gives us the sampling distribution which we are seeking. We use this method in the following theorems:

(29) Let $x_1 \sim N(\mu_1, \sigma_1^2)$ and $x_2 \sim N(\mu_2, \sigma_2^2)$ be two mutually independent normal variates. Then their weighted sum $y = \beta_1 x_1 + \beta_2 x_2$ is a normal variate $y \sim N(\mu_1\beta_1 + \mu_2\beta_2, \sigma_1^2\beta_1^2 + \sigma_2^2\beta_2^2)$.

Proof. From the previous theorem it follows that

$$(30)\qquad \begin{aligned} M(y,t) &= E\{e^{(\beta_1 x_1 + \beta_2 x_2)t}\} \\ &= \int_{x_1}\int_{x_2} e^{\beta_1 x_1 t} e^{\beta_1 x_1 t} f(x_1, x_2)dx_1 dx_2 \\ &= \int_{x_1} e^{\beta_1 x_1 t} f(x_1)dx_1 \int_{x_2} e^{\beta_1 x_1 t} f x_2)dx_2, \end{aligned}$$

since $f(x_1,x_2)=f(x_1)f(x_2)$ by the assumption of mutual independence. Thus

$$M(y,t)=E(e^{\beta_1x_1t})E(e^{\beta_2x_2t}). \tag{31}$$

But, if x_1 and x_2 are normal, then

$$E(e^{\beta_1x_1t})=e^{\mu_1\beta_1t}e^{(\sigma_1\beta_1t)^2/2} \quad\text{and}\quad E(e^{\beta_2x_2t})=e^{\mu_2\beta_2t}e^{(\sigma_2\beta_2t)^2/2}. \tag{32}$$

Therefore

$$M(y,t)=e^{(\mu_1\beta_1+\mu_2\beta_2)t}e^{(\sigma_1\beta_1+\sigma_2\beta_2)^2t^2/2},$$

which signifies that $y\sim N(\mu_1\beta_1+\mu_2\beta_2,\sigma_1^2\beta_1^2+\sigma_2^2\beta_2^2)$.

This theorem may be generalised immediately to encompass a weighted combination of an arbitrary number of mutually independent normal variates.

Next we state a theorem which indicates that the sum of a set of independent chi-square variates is itself a chi-square variate:

(33) Let $x_1,x_2,\ldots,x_n$ be n mutually independent chi-square variates with $x_i\sim\chi^2(k_i)$ for all i. Then $y=\sum_{i=1}^n x_i$ has the chi-square distribution with $k=\sum k_i$ degrees of freedom. That is to say, $y\sim\chi^2(k)$.

Proof. We have

$$\begin{aligned}M(y,t)&=E\{e^{(x_1+x_2+\cdots+x_n)t}\}\\&=E\{e^{x_1t}\}E\{e^{x_2t}\}\cdots E\{e^{x_nt}\},\end{aligned} \tag{34}$$

since $x_1,\ldots,x_n$ are mutually independent. But we know from (12) that a $\chi^2(r)$ variate has the distribution of a $\gamma_2(r/2,2)$ variate and that we know that the moment generating function of the $\chi^2(r)$ is $(1-2t)^{-(r/2)}$. Hence

$$\begin{aligned}M(y,t)&=(1-2t)^{-(k_1/2)}(1-2t)^{-(k_2/2)}\cdots(1-2t)^{-(k_n/2)}\\&=(1-2t)^{-(k/2)}.\end{aligned} \tag{35}$$

Therefore $y\sim\chi^2(k)$.

We have already shown that, if $x\sim N(\mu,\sigma^2)$, then $\{(x-\mu)/\sigma\}^2\sim\chi^2(1)$. From this it follows, in view of the preceding result, that

(36) If $\{x_1,x_2,\ldots,x_n\}$ is a random sample with $x_i\sim N(\mu,\sigma^2)$ for all i, then

$$y=\sum_{i=1}^n\left(\frac{x_i-\mu}{\sigma}\right)^2$$

has a chi-square distribution with n degrees of freedom, which can be expressed by writing $y\sim\chi^2(n)$.

The Decomposition of a Chi-Square

Consider the following identity:

$$\begin{aligned}\sum_{i=1}^{n}(x_i-\mu)^2&=\sum_{i=1}^{n}\{(x_i-\bar{x})-(\mu-\bar{x})\}^2\\&=\sum(x_i-\bar{x})^2-2\sum(x_i-\bar{x})(\mu-\bar{x})+n(\mu-\bar{x})^2\\&=\sum(x_i-\bar{x})^2+n(\mu-\bar{x})^2.\end{aligned}\tag{37}$$

We can show that if $x_1,\ldots,x_n$ constitute a random sample for a normal distribution, then the variables $(x_i-\bar{x}); i=1,\ldots,n$ are statistically independent of the variable $(\bar{x}-\mu)$. This indicates that the two elements of the L.H.S of equation (25) are independent. We shall take the result on trust for the moment and we shall prove it later. Given this result, we are in the position to state an important result concerning the decomposition of a chi-square variate:

(38) Let $\{x_1,x_2,\ldots,x_n\}$ be a random sample with $x_i\sim N(\mu,\sigma^2)$ for all i. Then

$$\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{\sigma^2}=\sum_{i=1}^{n}\frac{(x_i-\bar{x})^2}{\sigma^2}+n\frac{(\bar{x}-\mu)^2}{\sigma^2}$$

is a sum of statistically independent terms with

(a) $$\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{\sigma^2}\sim\chi^2(n),$$

(b) $$\sum_{i=1}^{n}\frac{(x_i-\bar{x})^2}{\sigma^2}\sim\chi^2(n-1),$$

(c) $$n\frac{(\bar{x}-\mu)^2}{\sigma^2}\sim\chi^2(2).$$

Proof. In proving this, we recognise at the start that, if $x_i\sim N(\mu,\sigma^2)$ for all i, then

$$\begin{aligned}&\bar{x}\sim N(\mu,\sigma^2/n)\qquad\text{and}\qquad\sqrt{n}\frac{(\bar{x}-\mu)}{\sigma}\sim N(0,1),\\&\text{whence}\qquad n\frac{(\bar{x}-\mu)^2}{\sigma^2}\sim\chi^2(2).\end{aligned}\tag{39}$$

Thus we have the result under (36, c). Moreover, we have already established the result under (36, a); and so it remains to demonstrate the result under (36, b). Consider, therefore, the moment generating function of the $\chi^2(n)$ variate. This is

$$(40)\qquad \begin{aligned} E\left[\exp\left\{t\sum\frac{(x_i-\mu)^2}{\sigma^2}\right\}\right] &= E\left[\exp\left\{tn\frac{(\bar{x}-\mu)^2}{\sigma^2}+t\frac{\sum(x_i-\bar{x})^2}{\sigma^2}\right\}\right]\\ &= E\left[\exp\left\{tn\frac{(\bar{x}-\mu)^2}{\sigma^2}\right\}\right]E\left[\exp\left\{t\frac{\sum(x_i-\bar{x})^2}{\sigma^2}\right\}\right]. \end{aligned}$$

Here the statistical independence of the components of the sum has allowed the expectation to be decomposed into the product of two expectations. Since the sum on the LHS and the first of the components of the RHS have been recognised as chi-square variates, we can make use of the know forms of the moment-generating functions to rewrite the equation as

$$(41)\qquad (1-2t)^{-n/2}=(1-2t)^{-1/2}E\left[\exp\left\{t\frac{\sum(x_i-\bar{x})^2}{\sigma^2}\right\}\right].$$

It follows that

$$(42)\qquad E\left[\exp\left\{t\frac{\sum(x_i-\bar{x})^2}{\sigma^2}\right\}\right]=\frac{(1-2t)^{-n/2}}{(1-2t)^{-1/2}}=(1-2t)^{-(n-1)/2};$$

and this is the moment generating function of a $\chi^2(n-1)$ variate. The result under (36, b) follows immediately.

The Independence of the Sample Mean and the Sample Variance.

(43) Let $\{x_1,x_2,\ldots,x_n\}$ be a random sample with $x_i\sim N(\mu,\sigma^2)$ for all i. Then

$$\bar{x}=\frac{1}{n}\sum_{i=1}^{n}x_i \qquad\text{and}\qquad \sum_{i=1}^{n}(x_i-\bar{x})^2$$

are statistically independent random variables.

Proof. To prove this proposition we shall adopt a matrix notation. Consider the summation vector $\mathbf{i}=[1,1,\ldots,1]'$ comprising n units.

We can use this to construct the matrix operator $P=\mathbf{i}(\mathbf{i}'\mathbf{i})^{-1}\mathbf{i}'=\mathbf{i}\mathbf{i}'/n$. If $\mathbf{x}=[x_1,x_2,\ldots,x_n]'$ is the vector of the sample elements, then $P\mathbf{x}=\bar{x}\mathbf{i}=[\bar{x},\bar{x},\ldots,\bar{x}]'$ is a vector which contain n repetitions of the sample mean. Also $(I-P)\mathbf{x}=\mathbf{x}-\bar{x}\mathbf{i}$ is the vector of the deviations of the sample elements from their mean. Observe that $P=P'=P^2$ and that, likewise, $I-P=(I-P)'=(I-P)^2$.

The matrix P is used in constructing the following identity:

$$(44)\qquad \begin{aligned} \mathbf{x}-\mu\mathbf{i} &= P(\mathbf{x}-\mu\mathbf{i})+(I-P)(\mathbf{x}-\mu\mathbf{i})\\ &= (\bar{x}-\mu)\mathbf{i}+(\mathbf{x}-\bar{x}\mathbf{i}). \end{aligned}$$

The quadratic product of these vectors is

$$\begin{aligned}(\mathbf{x}-\mu\mathbf{i})'(\mathbf{x}-\mu\mathbf{i}) &= (\mathbf{x}-\mu\mathbf{i})'P(\mathbf{x}-\mu\mathbf{i}) + (\mathbf{x}-\mu\mathbf{i})'(I-P)(\mathbf{x}-\mu\mathbf{i}) \\ &= n(\bar{x}-\mu)^2 + \sum_{i=1}^{n}(x_i-\bar{x})^2.\end{aligned} \tag{45}$$

We demonstrate the proposition by showing that this can be written as

$$(\mathbf{x}-\mu\mathbf{i})'(\mathbf{x}-\mu\mathbf{i}) = w_1^2 + \mathbf{w}_2'\mathbf{w}_2 \tag{46}$$

Where w_1 and $\mathbf{w}_2$ are mutually independent normal variates by virtue of having zero covariance. Consider therefore the matrix

$$C' = \begin{bmatrix} c_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & \cdots & 0 & 0 \\ \frac{1}{\sqrt{2.3}} & \frac{1}{\sqrt{2.3}} & \frac{-2}{\sqrt{2.3}} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{1}{\sqrt{(n-1)n}} & \frac{1}{\sqrt{(n-1)n}} & \frac{1}{\sqrt{(n-1)n}} & \cdots & & \frac{-(n-1)}{\sqrt{(n-1)n}} \end{bmatrix} \tag{47}$$

It can be seen that $C'C = CC' = I$ is the identity matrix. Moreover

$$\begin{aligned} CC' &= \mathbf{c}_1\mathbf{c}_1' + C_2C_2' \\ &= P + (I-P). \end{aligned} \tag{48}$$

Now if the vector $(\mathbf{x}-\mu\mathbf{i}) \sim N(0, \sigma^2 I)$ has a multivariate spherical normal distribution, which is to say that its elements are statistically independent, then the same must be true of the vector

$$C'(\mathbf{x}-\mu\mathbf{i}) = \begin{bmatrix} \mathbf{c}_1'(\mathbf{x}-\mu\mathbf{i}) \\ C_2'(\mathbf{x}-\mu\mathbf{i}) \end{bmatrix} = \begin{bmatrix} w_1 \\ \mathbf{w}_2 \end{bmatrix}. \tag{49}$$

Finally, we recognise that

$$\begin{aligned} w_1 = \mathbf{c}_1'(\mathbf{x}-\mu\mathbf{i}) = \sqrt{n}(\bar{x}-\mu) \qquad \text{and} \\ \mathbf{w}_2'\mathbf{w}_2 = (\mathbf{x}-\mu\mathbf{i})C_2C_2'(\mathbf{x}-\mu\mathbf{i}) = \sum_{i=1}^{n}(x_i-\bar{x})^2 \end{aligned} \tag{50}$$

must therefore be statistically independent, which proves the proposition.

Student's t Distribution and Fisher's F Distribution

Two distributions which are of prime importance in statistical inference are the t and the F

(51) Let $z \sim N(0,1)$ be a standard normal variate, and let $w \sim \chi^2(n)$ be a chi-square variate of n degrees of freedom distributed independently of z. Then

$$t = \left\{ z \bigg/ \sqrt{\frac{w}{n}} \right\} \sim t(n)$$

is said to have a t distribution of n degrees of freedom.

We shall endeavour to find the functional form of the $t(n)$ distribution. Consider the density functions of z and w which are respectively

$$(52) \qquad N(z;0,1) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \qquad \text{and} \qquad \chi^2(w;n) = \frac{e^{-w/2} w^{(n/2)-1}}{\Gamma(n/2)2^{n/2}}.$$

On account of the independence of z and w, we can write their joint density function as

$$(53) \qquad \psi(z,w) = \frac{1}{\sqrt{2\pi}} \frac{e^{-z^2/2} e^{-w/2} w^{(n/2)-1}}{\Gamma(n/2)2^{n/2}},$$

which is the product of the two density functions. We can proceed to find a joint density function $\gamma(t,w)$ by applying the change of variable technique to $\psi(z,w)$. Consider the transformation of $\{(z,w); -\infty < z < \infty; 0 < w < \infty\}$ into $\{(t,w); -\infty < t < \infty; 0 < w < \infty\}$. In view of the relationship $z = t\sqrt{w}/\sqrt{n}$, we derive the following Jacobian which is the determinant of the matrix of the derivatives of the inverse transformation mapping from (t,w) to (z,w):

$$(54) \qquad \begin{vmatrix} \dfrac{\partial z}{\partial t} & \dfrac{\partial z}{\partial w} \\ \dfrac{\partial w}{\partial t} & \dfrac{\partial w}{\partial w} \end{vmatrix} = \begin{vmatrix} \dfrac{\sqrt{w}}{\sqrt{n}} & \dfrac{t}{2\sqrt{wn}} \\ 0 & 1 \end{vmatrix} = \frac{\sqrt{w}}{\sqrt{n}}.$$

Therefore

$$(55) \qquad \begin{aligned} \gamma(t,w) &= \psi\{z(t,w),w\}|J| \\ &= \frac{1}{\sqrt{2\pi}} \frac{e^{-t^2 w/(2n)} e^{-w/2} w^{(n/2)-1}}{\Gamma(n/2)2^{n/2}} \frac{\sqrt{w}}{\sqrt{n}}. \end{aligned}$$

By simplifying, we get

$$(56) \qquad \begin{aligned} \gamma(t,w) &= \psi\{z(t,w),w\}|J| \\ &= \frac{w^{(n-1)/2}}{\sqrt{\pi n}\Gamma(n/2)2^{(n+1)/2}} \exp\left\{ -\frac{w}{2}\left(1 + \frac{t^2}{n}\right)\right\}. \end{aligned}$$

Let us define $q = w(1 + t^2/n)/2$. Then

$$(57) \qquad w = \frac{2q}{(1 + t^2/n)} \qquad \text{and} \qquad \frac{dw}{dq} = \frac{2}{(1 + t^2/n)},$$

and we may write

$$\begin{aligned} g(t) &= \int_0^\infty \gamma\{t, w(q)\} \frac{dw}{dq} dq \\ &= \int_0^\infty \frac{1}{\sqrt{\pi n}\Gamma(n/2)2^{(n+1)/2}} \left(\frac{2q}{1+t^2/n} \right)^{(n-1)/2} e^{-q} \left(\frac{2}{1+t^2/n} \right) dq \qquad (58) \\ &= \frac{1}{\sqrt{\pi n}\Gamma(n/2)} \frac{1}{(1+t^2/n)^{(n+1)/2}} \int_0^\infty e^{-q} q^{(n-1)/2} dq. \end{aligned}$$

But, by definition. the value of the integral is $\Gamma\{(n+1)/2\}$, so

$$(59) \qquad g(t) = \frac{\Gamma\{(n+1)/2\}}{\sqrt{\pi n}\Gamma(n/2)} \frac{1}{(1+t^2/n)^{(n+1)/2}}.$$

This gives us the functional form of Student's t distribution.