# THE POISSON DISTRIBUTION

The Poisson distribution is a limiting case of the binomial distribution which arises when the number of trials $n$ increases indefinitely whilst the product $\mu = np$, which is the expected value of the number of successes from the trials, remains constant. Consider the binomial probability mass function:

$$(1) \qquad b(x; n, p) = \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x}.$$

This can be rewritten as

$$(2) \qquad \frac{\mu^x}{x!} \frac{n!}{(n-x)!n^x} \left(1 - \frac{\mu}{n}\right)^n \left(1 - \frac{\mu}{n}\right)^{-x}.$$

The expression may be disassembled for the purpose of taking limits in the component parts. The limits in question are

$$(3) \qquad \lim(n \to \infty)\frac{n!}{(n-x)!n^x} = \lim(n \to \infty)\left\{\frac{n(n-1)\cdots(n-x+1)}{n^x}\right\}$$

$$= 1,$$

$$(4) \qquad \lim(n \to \infty)\left(1 - \frac{\mu}{n}\right)^n = e^{-\mu},$$

$$(5) \qquad \lim(n \to \infty)\left(1 - \frac{\mu}{n}\right)^{-x} = 1.$$

On reassembling the parts, it is found that the the binomial function has a limiting form of

$$(6) \qquad \lim(n \to \infty)b(x; n, p) = \frac{\mu^x e^{-\mu}}{x!}.$$

This is the Poisson function.

The Poisson function can be derived by considering a specification for a so-called emission process or an arrival process. One can imagine a Geiger counter which registers the impact of successive radioactive alpha particles upon a thin metallic film. Let $f(x, t)$ denote the probability of $x$ impacts or arrivals in the time interval $(0, t]$. The following conditions are imposed:

    (a) The probability of a single arrival in a very short time interval $(t, t + \Delta t]$ is $f(1, \Delta t) = a\Delta t$,

(b) The probability of more than one arrival during that time interval is negligible,

(c) The probability of an arrival during the time interval is independent of any occurrences in previous periods.

Certain consequences follow from these assumptions; and it can be shown that the Poisson distribution is the only distribution which fits the specification.

As a first consequence, it follows from the assumptions that the probability of there being $x$ arrivals in the interval $(0, t + \Delta t]$ is

(7)
$$f(x, t + \Delta t) = f(x, t)f(0, \Delta t) + f(x - 1, t)f(1, \Delta t)$$
$$= f(x, t)(1 - a\Delta t) + f(x - 1, t)a\Delta t.$$

This expression follows from the fact that there are two mutually exclusive ways in which the circumstance can arise. The first way is when all of the arrivals occur in the interval $(0, t]$ and none occur in the interval $(t, t + \Delta t]$. The second way is when $t - 1$ of the arrivals occur in the interval $(0, t]$ and one arrival occurs in the interval $(t, t + \Delta t]$. All other possibilities are ruled out by assumption (b). Assumption (c) implies that the probabilities of these two mutually exclusive or disjoint events are obtained by multiplying the probabilities of the events of the two sub-intervals.

The next step in the chain of deductions is to find the derivative of the function $f(x, t)$ with respect to $t$. From equation (7), it follows immediately that

(8)
$$\frac{df(x, t)}{dt} = \lim(\Delta t \to 0)\frac{f(x, t + \Delta t) - f(x, t)}{\Delta t}$$
$$= a\{f(x - 1, t) - f(x, t)\}.$$

The final step is to show that the function

(9)
$$f(x, t) = \frac{(at)^x e^{-at}}{x!}$$

satisfies the condition of equation (8). This is a simple matter of confirming that, according to the product rule of differentiation, we have

(10)
$$\frac{df(x, t)}{dt} = \frac{ax(at)^{x-1}e^{-at}}{x!} - \frac{a(at)^x e^{-at}}{x!}$$
$$= a\left\{\frac{(at)^{x-1}e^{-at}}{(x - 1)!} - \frac{(at)^x e^{-at}}{x!}\right\}$$
$$= a\{f(x - 1, t) - f(x, t)\}.$$

### Exercises

**1.** The probability density function governing the minutes of time $t$ spent waiting outside a telephone box is given by $f(t) = ae^{-at}$.

    (a) Determine the probability of having to wait for more than $t$ minutes.

    (b) Show that the probability of having to wait *more than* $2t$ minutes, given that you have waited for $t$, is the same as the unconditional probability of having to wait more than $t$ minutes.

What are the implications of this result?

**2.** Prove that the function $f(x) = \frac{1}{4}(\frac{3}{4})^x$; $x = 1, 2, \ldots$, constitutes a probability mass function. What is the probability that $x$ will assume any integer value from 0 to 3. Find the value of $n$ such that $P(x < n) = 0.9$ approximately.

**3.** Let $x$ be distributed as $f(x) = e^{-x}$ over the set of positive real numbers. Compute the probability that the random interval $(x, 3x)$ includes the point $x = 3$. What is the expected length of the interval?