# MOMENT GENERATING FUNCTIONS

**The natural number e.** The number $e = \{2.7183\ldots\}$ is defined by

$$e = \lim(n \to \infty) \left(1 + \frac{1}{n}\right)^n .$$

The binomial expansion indicates that

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^2 + \cdots .$$

Using this, we get

$$\left(1 + \frac{1}{n}\right)^n = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 + \cdots .$$

Taking limits as $n \to \infty$ of each term of the expansion gives

$$\lim(n \to \infty) \left(1 + \frac{1}{n}\right)^n = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = e.$$

**The expansion of $e^x$.** There is also

$$e^x = \lim(p \to \infty)\left(1 + \frac{1}{p}\right)^{px}$$
$$= \lim(n \to \infty)\left(1 + \frac{x}{n}\right)^{n}; n = px.$$

Using the binomial expansion in the same way as before, it can be shown that

$$e^x = \frac{x^0}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

Also

$$e^{xt} = 1 + xt + \frac{x^2t^2}{2!} + \frac{x^3t^3}{3!} + \cdots.$$

**The moment generating function.** The m.f.g. of the random variable $x \sim f(x)$ is a random is defined as

$$M(x,t) = E(e^{xt}) = 1 + tE(x) + \frac{t^2}{2!}E(x^2) + \frac{t^3}{3!}E(x^3) + \cdots.$$

Consider the following derivatives in respect of $t^r/r!$, which is the coefficient associated with $E(x^r)$ within t $M(x,t)$:

$$\frac{d}{dt}\left(\frac{t^r}{r!}\right) = \frac{rt^{r-1}}{r!} = \frac{t^{r-1}}{(r-1)!}$$

$$\frac{d^2}{dt^2}\left(\frac{t^r}{r!}\right) = \frac{r(r-1)t^{r-2}}{r!} = \frac{t^{r-2}}{(r-2)!}$$

$$\vdots$$

$$\frac{d^r}{dt^r}\left(\frac{t^r}{r!}\right) = \frac{\{r(r-1)\cdots 2.1\}t^0}{r!} = 1$$

$$\vdots$$

$$\frac{d^k}{dt^k}\left(\frac{t^r}{r!}\right) = 0 \quad \text{for} \quad k > r.$$

From this, it follows that

$$\frac{d^r}{dt^r}M(x,t) = \frac{d^r}{dt^r}\left\{\sum_{h=0}^{\infty}\frac{t^h}{h!}E(x^h)\right\}$$
$$= E(x^r) + tE(x^{r+1}) + \frac{t^2}{2!}E(x^{r+2}) + \frac{t^3}{3!}E(x^{r+3}) + \cdots.$$

To eliminate the terms in higher powers of $x$, we set $t = 0$:

$$\left.\frac{d^r}{dt^r}M(x,t)\right|_{t=0} = E(x^r).$$

**Example.** Consider $x \sim f(x) = e^{-x}; 0 \leq x < \infty$. The corresponding moment generating function is

$$\begin{aligned} M(x,t) = E(e^{xt}) &= \int_0^\infty e^{xt} e^{-x} dx \\ &= \int_0^\infty e^{-x(1-t)} dx = \left[ -\frac{e^{-x(1-t)}}{1-t} \right]_0^\infty = \frac{1}{1-t}. \end{aligned}$$

Expanding this gives

$$M(x,t) = 1 + t + t^2 + t^3 + \cdots.$$

Differentiating successively and setting $t = 0$ gives

$$\begin{aligned} \frac{d}{dt} M(x,t) &= 1 + 2t + 3t^2 + \cdots \Big|_{t=0} = 1, \\ \frac{d^2}{dt^2} M(x,t) &= 2 + 6t + 12t^2 + \cdots \Big|_{t=0} = 2, \\ \frac{d^3}{dt^3} M(x,t) &= 6 + 24t + 60t^2 + \cdots \Big|_{t=0} = 6. \end{aligned}$$

This gives is $E(x)$, which has already been established via a direct approach. The variance may be found via the formula $V(x) = E(x^2) - \{E(x)\}^2 = 2 - 1 = 1$.

**The m.g.f of the Binomial Distribution.** Consider the binomial function

$$b(x;n,p) = \frac{n!}{x!(n-x)!}p^x q^{n-x} \quad \text{with} \quad q = 1-p.$$

Then the moment generating function is given by

$$\begin{aligned} M(x,t) &= \sum_{x=0}^{n} e^{xt} \frac{n!}{x!(n-x)!} p^x q^{n-x} \\ &= \sum_{x=0}^{n} \frac{n!}{x!(n-x)!} (pe^t)^x q^{n-x} \\ &= (pe^t + q)^n, \end{aligned}$$

where the final equality is understood by recognising that it represents the expansion of a binomial. If we differentiate the moment generating function with respect to $t$ using the function-of-a-function rule, then we get

$$\begin{aligned} \frac{dM(x,t)}{dt} &= n(q + pe^t)^{n-1} pe^t \\ &= npe^t (pe^t + q)^{n-1}. \end{aligned}$$

Evaluating this at $t = 0$ gives

$$E(x) = np(p+q)^{n-1} = np.$$

Notice that this result is already familiar and that we have obtained it previously by somewhat simpler means.

**The variance of the binomial.** The moment generating function of the binomial is

$$M(x,t) = (pe^t + q)^n,$$

and its first derivative is

$$\frac{dM(x,t)}{dt} = n(q + pe^t)^{n-1}pe^t = npe^t(pe^t + q)^{n-1}.$$

To find the second moment, we use the product rule

$$\frac{duv}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

to get

$$\begin{aligned}\frac{d^2M(x,t)}{dt^2} &= npe^t\big\{(n-1)(pe^t + q)^{n-2}pe^t\big\} + (pe^t + q)^{n-1}\big\{npe^t\big\}\\ &= npe^t(pe^t + q)^{n-2}\big\{(n-1)pe^t + (pe^t + q)\big\}\\ &= npe^t(pe^t + q)^{n-2}\big\{q + npe^t\big\}.\end{aligned}$$

Evaluating this at $t = 0$ gives

$$\begin{aligned}E(x^2) &= np(p + q)^{n-2}(np + q)\\ &= np(np + q).\end{aligned}$$

From this, we see that

$$\begin{aligned}V(x) &= E(x^2) - \big\{E(x)\big\}^2\\ &= np(np + q) - n^2p^2\\ &= npq.\end{aligned}$$

**Moments about the mean.** The moments about the mean of a random variable are generated by the function

$$M(x-\mu,t)=\exp\{-\mu t\}M(x,t).$$

To understood this, consider the identity

$$M(x-\mu,t)=E\big[\exp\{(x-\mu)t\}\big]=e^{-\mu t}E(e^{xt})=\exp\{-\mu t\}M(x,t).$$

For the binomial, the m.g.f. about the mean is

$$\begin{aligned}M(x-\mu,t)&=e^{-npt}(pe^t+q)^n=(pe^te^{-pt}+qe^{-pt})^n\\&=(pe^{qt}+qe^{-pt})^n.\end{aligned}$$

Differentiating this once gives

$$\frac{dM(x-\mu,t)}{dt}=n(pe^{qt}+qe^{-pt})^{n-1}(qpe^{qt}-pqe^{-pt}).$$

Differentiating again via the product rule gives

$$\frac{d^2M(x-\mu,t)}{dt^2}=u(q^2pe^{qt}+p^2qe^{-pt})+v\frac{du}{dt},$$

where

$$u(t)=n(pe^{qt}+qe^{-pt})\quad\text{and}$$

$$v(t)=(qpe^{qt}-pqe^{-pt}).$$

At $t=0$, these become $u(0)=n$ and $v(0)=0$, from which

$$V(x)=n(p^2q+q^2p)=npq(p+q)=npq.$$

**The moment generating function of a sum of independent random variables.** An important theorem concerns the moment generating function of a sum of independent random variables:

If $x \sim f(x)$ and $y \sim f(y)$ are independently distributed random variables with moment generating functions $M(x,t)$ and $M(y,t)$, then their sum $z = x+y$ has the moment generating function $M(z,t) = M(x,t)M(y,t)$.

This result follows from the fact that the independence of $x$ and $y$ implies that their joint probability density function $f(x,y) = f(x)f(y)$ is the product of their marginal density functions. From this, it follows that

$$\begin{aligned} M(x+y,t) &= \int_x \int_y e^{(x+y)t} f(x,y)dydx \\ &= \int_x e^{xt} f(x)dx \int_y e^{yt} f(y)dy \\ &= M(x,t)M(y,t). \end{aligned}$$

## THE POISSON DISTRIBUTION

The Poisson distribution is derived by letting the number of trials $n$ of the binomial distribution increase indefinitely while the product $\mu = np$ remains constant. Consider

$$b(x; n, p) = \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x}.$$

By setting $p = \mu/n$, this can be rewritten as

$$\frac{\mu^x}{x!} \frac{n!}{(n-x)!n^x} \left(1 - \frac{\mu}{n}\right)^n \left(1 - \frac{\mu}{n}\right)^{-x}.$$

The expression may be disassembled for the purpose of taking limits in the component parts:

$$\lim(n \to \infty) \frac{n!}{(n-x)!n^x} = \lim(n \to \infty) \left\{ \frac{n(n-1)\cdots(n-x+1)}{n^x} \right\} = 1,$$

$$\lim(n \to \infty) \left(1 - \frac{\mu}{n}\right)^n = e^{-\mu},$$

$$\lim(n \to \infty) \left(1 - \frac{\mu}{n}\right)^{-x} = 1.$$

On reassembling the parts, it is found that the binomial function has a limiting form of

$$\lim(n \to \infty) b(x; n, p) = \frac{\mu^x e^{-\mu}}{x!}.$$

This is the Poisson function.

**Poisson arrivals.** Consider, for example, the impact of alpha particles upon a Geiger counter. Let $f(x,t)$ denote the probability of $x$ impacts in the time interval $(0,t]$. The following conditions are imposed:

(a) The probability of a single impact in a very short time interval $(t, t+\Delta t]$ is $f(1,\Delta t) = a\Delta t$,

(b) The probability of more than one impact during that time interval is negligible,

(c) The probability of an impact during the interval is independent of the impacts in previous periods.

Consider the event of $x$ impacts in the interval $(0, t+\Delta t]$. There are two possibilities:

(i) All of the impacts occur in the interval $(0,t]$ and none in the interval $(t, t+\Delta t]$.

(ii) There are $t-1$ impacts in the interval $(0,t]$ followed by one impact in the interval $(t, t+\Delta t]$.

The probability of $x$ impacts in the interval is therefore

$$\begin{aligned} f(x, t+\Delta t) &= f(x,t)f(0,\Delta t) + f(x-1,t)f(1,\Delta t) \\ &= f(x,t)(1-a\Delta t) + f(x-1,t)a\Delta t. \end{aligned}$$

Assumption (b) excludes all other possibilities. Assumption (c) implies that the probabilities of the mutually exclusive events of (i) and (ii) are obtained by multiplying the probabilities of the constituent events of the two sub-intervals.

By rearranging the equation

$$f(x,t+\Delta t) = f(x,t)(1-a\Delta t) + f(x-1,t)a\Delta t,$$

we get

$$\frac{f(x,t+\Delta t) - f(x,t)}{\Delta t} = a\{f(x-1,t) - f(x,t)\}.$$

So, by letting $\Delta t \to 0$, we find that

$$\frac{df(x,t)}{dt} = a\{f(x-1,t) - f(x,t)\}.$$

The final step is to show that the function

$$f(x,t) = \frac{(at)^x e^{-at}}{x!}$$

satisfies the above equation. Thus, according to the product rule of differentiation, we have

$$\begin{aligned}\frac{df(x,t)}{dt} &= \frac{ax(at)^{x-1}e^{-at}}{x!} - \frac{a(at)^x e^{-at}}{x!} \\ &= a\left\{\frac{(at)^{x-1}e^{-at}}{(x-1)!} - \frac{(at)^x e^{-at}}{x!}\right\} \\ &= a\Big\{f(x-1,t) - f(x,t)\Big\}.\end{aligned}$$