MOMENT GENERATING FUNCTIONS

The natural number e. Consider the irrational number $e = \{2.7183\ldots\}$. This is defined by

$$e = \lim(n \to \infty) \left(1 + \frac{1}{n}\right)^n.$$

Consider the series expansion of the expression that is being taken to the limit. The binomial expansion indicates that

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^2 + \cdots.$$

Using this, we get

$$\left(1 + \frac{1}{n}\right)^n = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 + \cdots$$

Taking limits as $n \to \infty$ of each term of the expansion gives

$$\lim(n \to \infty)\left(1 + \frac{1}{n}\right)^n = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = e$$

There is also

$$e^x = \lim(p \to \infty)\left(1 + \frac{1}{p}\right)^{px} = \lim(n \to \infty)\left(1 + \frac{x}{n}\right)^n; n = px.$$

Using the binomial expansion in the same way as before, it can be shown that

$$e^x = \frac{x^0}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

Also

$$e^{xt} = 1 + xt + \frac{x^2t^2}{2!} + \frac{x^3t^3}{3!} + \cdots.$$

The moment generating function. Now imagine that $x \sim f(x)$ is a random variable, and let us define

$$M(x,t) = E(e^{xt}) = 1 + tE(x) + \frac{t^2}{2!}E(x^2) + \frac{t^3}{3!}E(x^3) + \cdots.$$

This is the *moment generating function* or *m.g.f.* of x. Each of its terms contains one of the moments of the random variable x taken about the origin.

The problem is to extract the relevant moment from this expansion. The method is a follows. To find the rth moment $E(x^r)$, differentiate $M(x,t) = E(e^{xt})$

r times in respect of t. Then, set t to zero and the moment drops out. To demonstrate this consider the following sequence of derivatives in respect of the coefficient associated with $E(x^r)$ within the series expansion of $M(x,t)$:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{t^r}{r!}\right) &= \frac{rt^{r-1}}{r!} \\
\frac{d^2}{dt^2}\left(\frac{t^r}{r!}\right) &= \frac{r(r-1)t^{r-2}}{r!} \\
&\vdots \\
\frac{d^r}{dt^r}\left(\frac{t^r}{r!}\right) &= \frac{\{r(r-1)\cdots 2.1\}t^0}{r!} = 1 \\
&\vdots \\
\frac{d^k}{dt^k}\left(\frac{t^r}{r!}\right) &= 0 \quad \text{for} \quad k > r.
\end{aligned}$$

Thus it follows that

$$\begin{aligned}
\frac{d^r}{dt^r}M(x,t) &= \frac{d^r}{dt^r}\left\{\sum_{h=0}^{\infty}\frac{t^h}{h!}E(x^h)\right\} \\
&= E(x^r) + tE(x^{r+1}) + \frac{t^2}{2!}E(x^{r+2}) + \frac{t^3}{3!}E(x^{r+3}) + \cdots.
\end{aligned}$$

Therefore, to eliminate the terms in higher powers of x, we set $t = 0$ to give

$$\frac{d^r}{dt^r}M(x,t)\bigg|_{t=0} = E(x^r).$$

Example. Consider $x \sim f(x) = e^{-x}; 0 \leq x < \infty$. The corresponding moment generating function is

$$\begin{aligned}
M(x,t) = E(e^{xt}) &= \int_0^{\infty} e^{xt}e^{-x}dx \\
&= \int_0^{\infty} e^{-x(1-t)}dx = \left[-\frac{e^{-x(1-t)}}{1-t}\right]_0^{\infty} = \frac{1}{1-t}.
\end{aligned}$$

Expanding this gives

$$M(x,t) = 1 + t + t^2 + t^3 + \cdots.$$

Differentiating successively and setting $t = 0$ gives

$$\begin{aligned}
\frac{d}{dt}M(x,t) &= 1 + 2t + 3t^2 + \cdots\bigg|_{t=0} = 1, \\
\frac{d^2}{dt^2}M(x,t) &= 2 + 6t + 12t^2 + \cdots\bigg|_{t=0} = 2, \\
\frac{d^3}{dt^3}M(x,t) &= 6 + 24t + 60t^2 + \cdots\bigg|_{t=0} = 6.
\end{aligned}$$

This gives is $E(x)$, which has already been established via a direct approach. The variance may be found via the formula $V(x) = E(x^2) - \{E(x)\}^2 = 2 - 1 = 1$.

The m.g.f of the Binomial Distribution. Consider the binomial function

$$b(x; n, p) = \frac{n!}{x!(n-x)!} p^x q^{n-x} \quad \text{with} \quad q = 1 - p.$$

Then the moment generating function is given by

$$\begin{aligned} M(x,t) &= \sum_{x=0}^{n} e^{xt} \frac{n!}{x!(n-x)!} p^x q^{n-x} \\ &= \sum_{x=0}^{n} \frac{n!}{x!(n-x)!} (pe^t)^x q^{n-x} \\ &= (pe^t + q)^n, \end{aligned}$$

where the final equality is understood by recognising that it represents the expansion of binomial. If we differentiate the moment generating function with respect to t using the function-of-a-function rule, then we get

$$\begin{aligned} \frac{dM(x,t)}{dt} &= n(q + pe^t)^{n-1} pe^t \\ &= npe^t (pe^t + q)^{n-1}. \end{aligned}$$

Evaluating this at $t = 0$ gives

$$E(x) = np(p + q)^{n-1} = np.$$

Notice that this result is already familiar and that we have obtained it previously by somewhat simpler means.

To find the second moment, we use the product rule

$$\frac{duv}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

to get

$$\begin{aligned} \frac{d^2M(x,t)}{dt^2} &= npe^t \big\{(n-1)(pe^t + q)^{n-2} pe^t\big\} + (pe^t + q)^{n-1} \big\{npe^t\big\} \\ &= npe^t (pe^t + q)^{n-2} \big\{(n-1)pe^t + (pe^t + q)\big\} \\ &= npe^t (pe^t + q)^{n-2} \big\{q + npe^t\big\}. \end{aligned}$$

Evaluating this at $t = 0$ gives

$$\begin{aligned} E(x^2) &= np(p + q)^{n-2}(np + q) \\ &= np(np + q). \end{aligned}$$

From this, we see that

$$\begin{aligned} V(x) &= E(x^2) - \big\{E(x)\big\}^2 \\ &= np(np+q) - n^2p^2 \\ &= npq. \end{aligned}$$

Theorems Concerning Moment Generating Functions. In finding the variance of the binomial distribution, we have pursed a method which is more laborious than it need by. The following theorem shows how to generate the moments about an arbitrary datum which we may take to be the mean of the distribution.

> *The function which generates moments about the mean of a random variable is given by $M(x-\mu,t) = \exp\{-\mu t\}M(x,t)$ where $M(x,t)$ is the function which generates moments about the origin.*

This result is understood by considering the following identity:

$$M(x-\mu,t) = E\big\{\exp\{(x-\mu)t\}\big\} = e^{-\mu t}E(e^{xt}) = \exp\{-\mu t\}M(x,t).$$

For an example, consider once more the binomial function. The moment generating function about the mean is then

$$\begin{aligned} M(x-\mu,t) &= e^{-npt}(pe^t+q)^n \\ &= (pe^te^{-pt}+qe^{-pt})^n \\ &= (pe^{qt}+qe^{-pt})^n. \end{aligned}$$

Differentiating this once gives

$$\frac{dM(x-\mu,t)}{dt} = n(pe^{qt}+qe^{-pt})^{n-1}(qpe^{qt}-pqe^{-pt}).$$

At $t=0$, this has the value of zero, as it should. Differentiating a second time according to the product rule gives

$$\frac{d^2M(x-\mu,t)}{dt^2} = u(q^2pe^{qt}+p^2qe^{-pt}) + v\frac{du}{dt},$$

where

$$\begin{aligned} u(t) &= n(pe^{qt}+qe^{-pt}) \quad \text{and} \\ v(t) &= (qpe^{qt}-pqe^{-pt}). \end{aligned}$$

At $t=0$ these become $u(0)=n$ and $v(0)=0$. It follows that

$$V(x) = n(p^2q+q^2p) = npq(p+q) = npq,$$

as we know from a previous derivation.

Another important theorem concerns the moment generating function of a sum of independent random variables:

If $x \sim f(x)$ and $y \sim f(y)$ be two independently distributed random variables with moment generating functions $M(x,t)$ and $M(y,t)$, then their sum $z = x + y$ has the moment generating function $M(z,t) = M(x,t)M(y,t)$.

This result is a consequence of the fact that the independence of x and y implies that their joint probability density function is the product of their individual marginal probability density functions: $f(x,y) = f(x)f(y)$. From this, it follows that

$$\begin{aligned} M(x+y,t) &= \int_x \int_y e^{(x+y)t} f(x,y)dydx \\ &= \int_x e^{xt} f(x)dx \int_y e^{yt} f(y)dy \\ &= M(x,t)M(y,t). \end{aligned}$$

The Poisson Distribution. The Poisson distribution is a limiting case of the binomial distribution which arises when the number of trials n increases indefinitely whilst the product $\mu = np$, which is the expected value of the number of successes from the trials, remains constant. Consider the binomial probability mass function:

$$b(x;n,p) = \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x}.$$

This can be rewritten as

$$\frac{\mu^x}{x!} \frac{n!}{(n-x)!n^x} \left(1 - \frac{\mu}{n}\right)^n \left(1 - \frac{\mu}{n}\right)^{-x}.$$

The expression may be disassembled for the purpose of taking limits in the component parts. The limits in question are

$$\begin{aligned} \lim(n \to \infty) \frac{n!}{(n-x)!n^x} &= \lim(n \to \infty) \left\{ \frac{n(n-1)\cdots(n-x+1)}{n^x} \right\} \\ &= 1, \end{aligned}$$

$$\lim(n \to \infty) \left(1 - \frac{\mu}{n}\right)^n = e^{-\mu},$$

$$\lim(n \to \infty) \left(1 - \frac{\mu}{n}\right)^{-x} = 1.$$

On reassembling the parts, it is found that the the binomial function has a limiting form of

$$\lim(n \to \infty) b(x;n,p) = \frac{\mu^x e^{-\mu}}{x!}.$$

This is the Poisson function.

The Poisson function can be derived by considering a specification for a so-called emission process or an arrival process. One can imagine a Geiger counter which registers the impact of successive radioactive alpha particles upon a thin metallic film. Let $f(x,t)$ denote the probability of x impacts or arrivals in the time interval $(0,t]$. The following conditions are imposed:

(a) The probability of a single arrival in a very short time interval $(t, t+\Delta t]$ is $f(1,\Delta t) = a\Delta t$,

(b) The probability of more than one arrival during that time interval is negligible,

(c) The probability of an arrival during the time interval is independent of any occurrences in previous periods.

Certain consequences follow from these assumptions; and it can be shown that the Poisson distribution is the only distribution which fits the specification.

As a first consequence, it follows from the assumptions that the probability of there being x arrivals in the interval $(0, t+\Delta t]$ is

$$\begin{aligned} f(x,t+\Delta t) &= f(x,t)f(0,\Delta t) + f(x-1,t)f(1,\Delta t) \\ &= f(x,t)(1-a\Delta t) + f(x-1,t)a\Delta t. \end{aligned}$$

This expression follows from the fact that there are two mutually exclusive ways in which the circumstance can arise. The first way is when all of the arrivals occur in the interval $(0,t]$ and none occur in the interval $(t,t+\Delta t]$. The second way is when $t-1$ of the arrivals occur in the interval $(0,t]$ and one arrival occurs in the interval $(t,t+\Delta t]$. All other possibilities are ruled out by assumption (b). Assumption (c) implies that the probabilities of these two mutually exclusive or disjoint events are obtained by multiplying the probabilities of the events of the two sub-intervals.

The next step in the chain of deductions is to find the derivative of the function $f(x,t)$ with respect to t. From the equation above, it follows immediately that

$$\begin{aligned} \frac{df(x,t)}{dt} &= \lim(\Delta t \to 0)\frac{f(x,t+\Delta t) - f(x,t)}{\Delta t} \\ &= a\bigl\{f(x-1,t) - f(x,t)\bigr\}. \end{aligned}$$

The final step is to show that the function

$$f(x,t) = \frac{(at)^x e^{-at}}{x!}$$

satisfies the condition of the above equation. This is a simple matter of confirming that, according to the product rule of differentiation, we have

$$\begin{aligned} \frac{df(x,t)}{dt} &= \frac{ax(at)^{x-1}e^{-at}}{x!} - \frac{a(at)^x e^{-at}}{x!} \\ &= a\left\{\frac{(at)^{x-1}e^{-at}}{(x-1)!} - \frac{(at)^x e^{-at}}{x!}\right\} \\ &= a\Bigl\{f(x-1,t) - f(x,t)\Bigr\}. \end{aligned}$$