1. Give an account of the axioms of Boolean algebra and, in the process, compare them with the axioms of arithmetic. Prove the rule of De Morgan that asserts that $A^c \cap B^c = (A \cup B)^c$, where the suffix c denotes complemetation.

A student has a probability of 0.2 of forgetting the book that he must return, a probability of 0.35 of forgetting his library card and a probability of forgetting both of 0.3.

(a) What is the probability that today he will have both the book and the library card in his possession

(b) Given that he has the book, what is the probability that he will find the library card?

Answer. We must prove that (i) $(A^c \cap B^c) \cup (A \cup B) = S$ and that (ii) $(A^c \cap B^c) \cap (A \cup B) = \emptyset$.

(i) There is $(A^c \cap B^c) \cup (A \cup B) = \{(A^c \cap B^c) \cup A\} \cup \{(A^c \cap B^c) \cup B\}$. But $\{(A^c \cap B^c) \cup A\} = \{([A^c \cup A] \cap [B^c \cup A])\} = B^c \cup A$ and there is $(A^c \cap B^c) \cup A = A^c \cup B$, whence the union of the two is $(B^c \cup A) \cup (A^c \cup B) = S$

(ii) There is $(A^c \cap B^c) \cap (A \cup B) = \{(A^c \cap B^c) \cap A\} \cup \{(A^c \cap B^c) \cap B\} = \emptyset \cap \emptyset$.

(a) Let A denote the event of forgetting the library card and let B dentote the event of forgetting the book. The event of forgetting neither is $A^c \cap B^c = (A \cup B)^c$. There is $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.35 + 0.2 - 0.3 = 0.25$. Therefore $P(A^c \cap B^c) = P\{(A \cup B)^c\} = 1 - P(A \cup B) = 0.75$.

(b) The probability of finding the library card given that he has found the book is $P(A^c|B^c) = P(A^c \cap B^c)/P(B^c) = 0.75/(1 - 0.2) = 15/16$.

2. Let $H_1, H_2, \ldots, H_k$ denote an exhaustive set of mutually exclusive hypotheses representing the possible causes of an event E. Show that

$$P(H_i|E) = \frac{P(E|H_i)P(H_i)}{P(E)},$$

where $P(E) = \sum_i P(E|H_i)P(H_i)$.

I always leave the house for work before my wife does. My wife takes the car on 3 out of the 5 workdays. Today, I have reckoned that there is an equal probabilty that either of us will be the first to return home. On returning from work, I have seen that the car is parked outside the house. How should I re-assess the probability that my wife has already returned home?

Answer. Let W be the circumstance of my wife's having returned home and let W^c be the circumstance that she has not returned. Let C signify the presence of the car outside the home.

The prior probability of my wife's being at home is $P(W) = 1/2$. The likelihood of car being parked outside, given that my wife is not at home, is $P(C|W^c) = 2/5$. Also, there is $P(C|W) = 1$, which is to say that it is certain that the car will be there if she is at home.

According to the theorem of Bayes, there is

$$P(W|C) = \frac{P(C|W)P(W)}{P(C)}, \quad \text{with}$$
$$P(C) = P(C|W)P(W) + P(C|W^c)P(W^c)$$

We calculate that

$$P(C) = \left(1 \times \frac{1}{2}\right) + \left(\frac{2}{5} \times \frac{1}{2}\right) = \frac{7}{10}$$

and that

$$P(W|C) = \left(1 \times \frac{1}{2}\right) \times \frac{10}{7} = \frac{5}{7}.$$

3. Derive, from first principles, the function expressing the probability of obtaining x successes in n independent trials when the probability of a success in any trial is p.

Experience suggests that 10% of of those making reservations at a restaurant will not arrive. The restaurant has eight tables, but it has taken ten reservations. What is the probability that it will be able to accomodate everyone who arrives?

Answer: The probability that x out of 10 rservations will materialise is given

$$b\left(x; n = 10, p = \frac{9}{10}\right) = \frac{10!}{(10-x)!x!}\left(\frac{9}{10}\right)^x \left(\frac{1}{10}\right)^{10-x}.$$

The probability the restaurant will be able to acommodate everyone is

$$\begin{aligned} 1 - b(9) - b(10) &= 1 - 10\left(\frac{9}{10}\right)^9 \left(\frac{1}{10}\right) - \left(\frac{9}{10}\right)^{10} \\ &= 1 - \left(\frac{9}{10}\right)^9 \left(\frac{19}{10}\right). \end{aligned}$$

4. Let x and y be random variables, and imagine that theconditional expectation of y given x is determined by the linearfunction $E(y|x) = \alpha + \beta x$. Find expressions for α and β that are interms of the moments of the joint distribution of x and y.

In a skiing competition, a competitor's overall time is found by adding his time in the downhill section to his time in the slalom section. Downhill times have an expected value of 1 minute and 15 seconds with a standard deviation of 5 seconds. Slalom times have an expected value of 1 minute and 45 seconds with a standard deviation of 6 seconds. Overall times have a standard deviation of 11 seconds. What is the expected overall time of a competitor who has recorded 1 minute and 10 seconds in the downhill section?

Answer. The downhill time is x, the slalom time is y.We have

$$\begin{aligned}
&E(x) = 75, \quad V(x) = 25,\\
&E(y) = 105, \quad V(y) = 36, \quad \text{and}\\
&V(x+y) = V(x) + V(y) + 2C(x,y) = 121.
\end{aligned}$$

Hence

$$C(x,y) = \frac{121 - 36 - 25}{2} = 30, \quad \text{and} \quad \beta = \frac{C(x,y)}{V(x)} = \frac{30}{25}.$$

If $x = 70$, then

$$\begin{aligned}
E(y|x) &= E(y) + \beta\{x - E(x)\}\\
&= 105 + \frac{30}{25}\{70 - 75\} = 99.
\end{aligned}$$

Also

$$E(x + y|x) = x + E(y|x) = 70 + 99 \quad \text{i.e.} \quad \text{2mins. 49secs.}$$

5. Describe (a) the difference between a one-tailed test of a statistical hypothesis and a two-tailed test, and (b) the difference between a Type I error and a Type II error.

Two guns in the same emplacement are firing in the same direction. Gun A is firing with a range of 5 km with a standard deviation of 200 m and gun B is firing with a range of 6 km with a standard derivation of 400 m. A shell falls 5.4 km from the emplacement. Test the hypothesis that it is fired by gun A, using a 5% level of significance, and evaluate the probability of committing a Type II error.

Answer. Let $H_A : x \sim N(\mu_A = 5, \sigma_A = 0.2)$ denote the hypothesis that the shell is from gun A and let $H_B : x \sim N(\mu_B = 6, \sigma_B = 0.4)$ denote the hypothesis that the shell is from gun B. Under H_A, there is $z_A = (x - \mu_A)/\sigma_A \sim N(0, 1)$ and, with $x = 5.4$, there is $z_A = (5.4 - 5)/0.2 = 0.2$, which is a significant value given that $P(z_A > c = 1.65) = 0.05$. Therefore, we might be prepared to reject H_A.

The type II error is that of accepting the null hypothesis H_A when the alternative hypothesis H_B is true. The null hypothesis is accepted whenever $(x - \mu_A)/\sigma_A < c$ or, equivalently, whenever $x < c\sigma_A + \mu_A = c^*$.

Under H_B, there is $P(x < c^*) = P\{(x - \mu_B)/\sigma_B = z_B < (c^* - \mu_B)/\sigma_B = (c\sigma_A + \mu_A - \mu_B)/\sigma_B\}$, where $z_B \sim N(0, 1)$. We find that

$$\frac{(c\sigma_A + \mu_A - \mu_B)}{\sigma_B} = \frac{1.65 \times 0.2 - 0.6}{0.4} = -0.675$$

Therefore, the probability of a Type II error is $P(z_b < -0.675) \simeq 0.25$.

6. Demonstrate how the moments of a random variable x may be obtained from its moment generating function by showing that the rth derivative of $E(e^{xt})$ with respect to t gives the value of $E(x^r)$ at the point where $t = 0$.

Demonstrate that the moment generating function of a sum of independent variables is the product of their individual moment generating functions.

Find the moment generating function of the point binomial

$$f(x; p) = p^x (1 - p)^{1-x}$$

where $x = 0, 1$. What is the relationship between this and the m.g.f. of the binomial distribution?

Find the variance of $x_1 + x_2$ when $x_1 \sim f(p_1 = 0.25)$ and $x_2 \sim f(p_2 = 0.75)$ are independent point binomials.

Answer. It is straightforward to show that $V(x_1) = V(x_2) = pq$ where $p = p_1$ and $q = p_2$ are used for ease of notation. Since x_1, x_2 are statistically independent, it follows that $V(x_1 + x_2) = V(x_1) + V(x_2) = 2pq$.

Alternatively, we may consider the moment generating functions of the two variables x_1 and x_2 which are respectively

$$M_1 = (p + qe^t) \qquad \text{and} \qquad M_2 = (q + pe^t).$$

The moment generating function of their sum $y = x_1 + x_2$ is

$$M = M_1 M_2 = (p + qe^t)(q + pe^t) = pq + q^2 e^t + p^2 e^t + pqe^{2t}.$$

Its first and second derivatives are

$$\frac{dM}{dt} = q^2e^t + p^2e^t + 2pqe^{2t},$$
$$\frac{d^2M}{dt^2} = q^2e^t + p^2e^t + 4pqe^{2t};$$

and setting $t = 0$ gives the following moments:

$$E(y) = q^2 + p^2 + 2pq = 1,$$
$$E(y^2) = q^2 + p^2 + 4pq = 1 + 2pq.$$

It follows that

$$V(y) = E(y^2) - \left\{E(y^2)\right\} = 2pq = 0.375$$