EXERCISES IN STATISTICS

Series A, No. 5

1. Find the moment generating function of $x \sim f(x) = 1$, where $0 < x < 1$, and thereby confirm that $E(x) = \frac{1}{2}$ and $V(x) = \frac{1}{12}$.

Answer: The moment generating function is

$$\begin{aligned} M(x,t) = E(e^{xt}) &= \int_0^1 e^{xt} dx \\ &= \left[\frac{e^{xt}}{t} \right]_0^1 = \frac{e^t}{t} - \frac{1}{t}. \end{aligned}$$

But

$$e^t = \frac{t^0}{0!} + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots,$$

so

$$\begin{aligned} M(x,t) &= \left[\frac{1}{t} + 1 + \frac{t}{2!} + \frac{t^2}{3!} + \frac{t^3}{4!} + \cdots \right] - \frac{1}{t} \\ &= 1 + \frac{t}{2} + \frac{t^2}{6} + \frac{t^3}{24} + \cdots. \end{aligned}$$

By the process of differentiating $M(x,t)$ with respect to t and the setting $t = 0$, we get

$$\begin{aligned} E(x) &= \left. \frac{\partial M(x,t)}{\partial t} \right|_{t=0} = \left[\frac{1}{2} + \frac{2t}{3!} + \frac{3t^2}{4!} + \cdots \right]_{t=0} = \frac{1}{2}, \\ E(x^2) &= \left. \frac{\partial^2 M(x,t)}{\partial t^2} \right|_{t=0} = \left[\frac{2}{3!} + \frac{6t}{4!} + \cdots \right]_{t=0} = \frac{1}{3}. \end{aligned}$$

Combining these results gives

$$V(x) = E(x^2) - \big\{E(x)\big\}^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

2. Find the moment generating function of $x \sim f(x) = ae^{-ax}; x \geq 0$.

Answer: The moment generating function is

$$\begin{aligned} M(x,t) = E(e^{xt}) &= \int_0^\infty e^{xt} a e^{-ax} dx = \int_0^\infty a e^{-x(a-t)} dx \\ &= \left[\frac{-ae^{-x(a-t)}}{a-t} \right]_0^\infty = \left[\frac{a}{a-t} \right] = \frac{1}{1 - t/a}. \end{aligned}$$

3. Prove that $x \sim f(x) = xe^{-x}; x \geq 0$ has a moment generating function of $1/(1-t)^2$. Hint: Use the change of variable technique to integrate with respect to $w = x(1-t)$ instead of x.

Answer: The moment generating function is

$$M(x,t) = E(e^{xt}) = \int_0^\infty e^{xt} xe^{-x} dx = \int_0^\infty xe^{-x(1-t)} dx.$$

Define $w = x(1-t)$. Then

$$x = \frac{w}{1-t} \quad \text{and} \quad \frac{dx}{dw} = \frac{1}{1-t}.$$

The change of variable technique indicates that

$$\int g(x)dx = \int g\{x(w)\}\frac{dx}{dw},$$

where $g(x) = xe^{-x(1-t)}$. Thus we find that

$$\begin{aligned} M(x,t) &= \int_0^\infty \frac{w}{1-t} e^{-w} \frac{1}{1-t} dw \\ &= \frac{1}{(1-t)^2} \int_0^\infty we^{-w} dw = \frac{1}{(1-t)^2}. \end{aligned}$$

Here the value of the final integral is unity, since the expression we^{-w}, which is to be found under the integral sign, has the same form as the p.d.f. of x.

To demonstrate directly that the value is unity, we can use the technique of integrating by parts. The formula is

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dv}dx.$$

Within the expression we^{-w}, we take $w = u$ and $e^{-w} = dv/dw$. Then we get

$$\begin{aligned} \int_0^\infty we^{-w} dw &= \left\{ \left[-we^{-w}\right]_0^\infty + \int_0^\infty e^{-w} dw \right\} \\ &= \int_0^\infty e^{-w} dw = \left[-e^{-w}\right]_0^\infty = 1. \end{aligned}$$

4. Using the theorem that the moment generating function of a sum of independent variables is the product of their individual moment generating functions, find the m.g.f. of $x_1 \sim e^{-x_1}$ and $x_2 \sim e^{-x_2}$ when $x_1, x_2 \geq 0$ are independent. Can you identify the p.d.f. of $f(x_1 + x_2)$ from this m.g.f.?

Answer: If x_1 and x_2 are independent, then their joint p.d.f. can be written as $f(x_1, x_2) = f(x_1)f(x_2)$; and it follows that

$$\begin{aligned} M(x_1 + x_2, t) &= \int_{x_2}\int_{x_1} e^{(x_1+x_2)t} f(x_1, x_2)dx_1 dx_2) \\ &= \int_{x_1} e^{x_1} f(x_1)dx_1 \int_{x_2} e^{x_2} f(x_2)dx_2 = M(x_1, t)M(x_2, t), \end{aligned}$$

or simply that $M(x_1 + x_2, t) = M(x_1, t)M(x_2, t)$. If x_1, x_2 are independent with $x_1 \sim e^{-x_1}$ and $x_2 \sim e^{-x_2}$, then

$$M(x_1, t) = M(x_2, t) = \frac{1}{1-t} \quad \text{and} \quad M(x_1 + x_2, t) = \frac{1}{(1-t)^2}.$$

But, according to the answer to question (3), this implies that

$$f(x_1 + x_2) = (x_1 + x_2)e^{-(x_1+x_2)}.$$

5. Find the moment generating function of the point binomial

$$f(x) = p^x(1-p)^{1-x}$$

where $x = 0, 1$. What is the relationship between this and the m.g.f. of the binomial distribution ?

Answer: If $f(x) = p^x(1-p)^{1-x}$ with $x = 0, 1$, then

$$\begin{aligned} M(x, t) &= \sum_{x=0,1} e^{xt} f(x) = e^0 p^0(1-p) + e^t p(1-p)^0 \\ &= (1-p) + pe^t = q + pe^t. \end{aligned}$$

But the binomial outcome $z = \sum_{i=1}^{n} x_i$ is the sum of n inepentent point-binomial outcomes; so it follows that the binomial m.g.f. is

$$M(z, t) = (q + pe^t)^n.$$