1. Find the moment generating function of $x \sim f(x) = 1$, where $0 < x < 1$, and thereby confirm that $E(x) = \frac{1}{2}$ and $V(x) = \frac{1}{12}$.

**Answer:** The moment generating function is

$$M(x, t) = E(e^{xt}) = \int_0^1 e^{xt}dx$$

$$= \left[ \frac{e^{xt}}{t} \right]_0^1 = \frac{e^t}{t} - \frac{1}{t}.$$

But

$$e^t = \frac{t^0}{0!} + \frac{t^1}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots,$$

so

$$M(x, t) = \left[ \frac{1}{t} + 1 + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots \right] - \frac{1}{t}$$

$$= 1 + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \cdots.$$

By the process of differentiating $M(x, t)$ with respect to $t$ and the setting $t = 0$, we get

$$E(x) = \frac{\partial M(x, t)}{\partial t} \bigg|_{t=0} = \left[ \frac{1}{2} + \frac{2t}{3!} + \frac{3t^2}{4!} + \cdots \right]_{t=0} = \frac{1}{2},$$

$$E(x^2) = \frac{\partial^2 M(x, t)}{\partial t^2} \bigg|_{t=0} = \left[ \frac{2}{3!} + \frac{6t}{4!} + \cdots \right]_{t=0} = \frac{1}{3}.$$

Combining these results gives

$$V(x) = E(x^2) - \{E(x)\}^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

2. Find the moment generating function of $x \sim f(x) = ae^{-ax}; x \geq 0$.

**Answer:** The moment generating function is

$$M(x, t) = E(e^{xt}) = \int_0^{\infty} e^{xt}ae^{-ax}dx = \int_0^{\infty} ae^{-(a-t)x}dx$$

$$= \left[ \frac{-ae^{-(a-t)x}}{a-t} \right]_0^{\infty} = \left[ \frac{a}{a-t} \right] = \frac{1}{1-t/a}.$$
3. Prove that \( x \sim f(x) = xe^{-x}; x \geq 0 \) has a moment generating function of \( 1/(1 - t)^2 \). Hint: Use the change of variable technique to integrate with respect to \( w = x(1 - t) \) instead of \( x \).

**Answer:** The moment generating function is

\[
M(x, t) = E(e^{xt}) = \int_0^\infty e^{xt}xe^{-x}dx = \int_0^\infty xe^{-x(1-t)}dx.
\]

Define \( w = x(1 - t) \). Then

\[
x = \frac{w}{1-t} \quad \text{and} \quad \frac{dx}{dw} = \frac{1}{1-t}.
\]

The change of variable technique indicates that

\[
\int g(x)dx = \int g\{x(w)\}\frac{dx}{dw},
\]

where \( g(x) = xe^{-x(1-t)} \). Thus we find that

\[
M(x, t) = \int_0^\infty \frac{w}{1-t}e^{-w} \frac{1}{1-t}dw
= \frac{1}{(1-t)^2} \int_0^\infty we^{-w}dw = \frac{1}{(1-t)^2}.
\]

Here the value of the final integral is unity, since the expression \( we^{-w} \), which is to be found under the integral sign, has the same form as the p.d.f. of \( x \).

To demonstrate directly that the value is unity, we can use the technique of integrating by parts. The formula is

\[
\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dv} dx.
\]

Within the expression \( we^{-w} \), we take \( w = u \) and \( e^{-w} = dv/dw \). Then we get

\[
\int_0^\infty we^{-w}dw = \left\{ [-we^{-w}]_0^\infty + \int_0^\infty e^{-w}dw \right\}
= \int_0^\infty e^{-w}dw = [-e^{-w}]_0^\infty = 1.
\]
4. Using the theorem that the moment generating function of a sum of independent variables is the product of their individual moment generating functions, find the m.g.f. of $x_1 \sim e^{-x_1}$ and $x_2 \sim e^{-x_2}$ when $x_1, x_2 \geq 0$ are independent. Can you identify the p.d.f. of $f(x_1 + x_2)$ from this m.g.f.?

**Answer:** If $x_1$ and $x_2$ are independent, then their joint p.d.f. can be written as $f(x_1, x_2) = f(x_1)f(x_2)$; and it follows that

$$M(x_1 + x_2, t) = \int_{x_2} \int_{x_1} e^{(x_1+x_2)t} f(x_1, x_2) dx_1 dx_2$$

$$= \int_{x_1} e^{x_1} f(x_1) dx_1 \int_{x_2} e^{x_2} f(x_2) dx_2 = M(x_1, t)M(x_2, t),$$

or simply that $M(x_1 + x_2, t) = M(x_1, t)M(x_2, t)$. If $x_1, x_2$ are independent with $x_1 \sim e^{-x_1}$ and $x_2 \sim e^{-x_2}$, then

$$M(x_1, t) = M(x_2, t) = \frac{1}{1 - t} \quad \text{and} \quad M(x_1 + x_2, t) = \frac{1}{(1 - t)^2}.$$

But, according to the answer to question (3), this implies that

$$f(x_1 + x_2) = (x_1 + x_2)e^{-(x_1+x_2)}.$$

5. Find the moment generating function of the point binomial

$$f(x) = p^x(1-p)^{1-x}$$

where $x = 0, 1$. What is the relationship between this and the m.g.f. of the binomial distribution?

**Answer:** If $f(x) = p^x(1-p)^{1-x}$ with $x = 0, 1$, then

$$M(x, t) = \sum_{x=0,1} e^{xt} f(x) = e^0 p^0(1-p) + e^t p(1-p)^0$$

$$= (1-p) + pe^t = q + pe^t.$$

But the binomial outcome $z = \sum_{i=1}^n x_i$ is the sum of $n$ independent point-binomial outcomes; so it follows that the binomial m.g.f. is

$$M(z, t) = (q + pe^t)^n.$$