EXERCISES IN STATISTICS

Series A, No. 4

1. Ten jurors are selected at random from a long list of names of which one third belong to women. The selection is regarded as unsatisfactory if either sex is represented by less than three people. Find the probability that the selection will be unsatisfactory.

Answer: The probability of a juror being a woman is 1/3. The number x of the women on the jury has a binomial distribution:

$$x \sim b\left(x, n = 10, p = \tfrac{1}{3}\right).$$

The selection is unsatisfactory if $x \in \{0, 1, 2, 8, 9, 10\}$, and the probability of this event is

$$\begin{aligned}
&b(0) + b(1) + b(2) + b(8) + b(9) + b(10) \\
&= \left(\frac{2}{3}\right)^{10} + 10\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{9} + \frac{10 \times 9}{2}\left(\frac{1}{3}\right)^{2}\left(\frac{2}{3}\right)^{8} \\
&\quad + \frac{10 \times 9}{2}\left(\frac{1}{3}\right)^{8}\left(\frac{2}{3}\right)^{2} + 10\left(\frac{1}{3}\right)^{9}\left(\frac{2}{3}\right) + \left(\frac{1}{3}\right)^{10} \\
&= \frac{17,856}{59,049} \simeq \frac{3}{10}.
\end{aligned}$$

2. Assume that the distribution of the heights of adult males is approximately normal with a mean of 69 ins and a standard deviation of 3 ins. What is the conditional probability that an individual will be taller than 72 ins if it is known that he is taller than 70 ins.?

Answer: The distribution of height is the normal distribution $N(\mu = 69, \sigma^2 = 9)$. We have to evaluate

$$P(x > 72 | x > 70) = \frac{P(x > 72)}{P(x > 70)}.$$

We begin by finding equivalent events in term of the standard normal variate $z = (x - \mu)/\sigma = (x - 69)/3$. We have the following equivalences:

$$x > 72 \iff z > \frac{72 - 69}{3} = 1,$$

$$x > 70 \iff z > \frac{702 - 69}{3} = \frac{1}{3}.$$

From tables of the standard normal, we find that

$$\begin{aligned} P(z > 1) &= 1 - (0.5 + 0.3413) \\ &= 0.1587. \end{aligned}$$

and that

$$\begin{aligned} P(z > \tfrac{1}{3}) &= 1 - (0.5 + 0.1293) \\ &= 0.3707. \end{aligned}$$

Therefore

$$P(x > 72 | x > 70) = \frac{P(x > 72)}{P(x > 70)}$$

$$= \frac{P(z > 1)}{P(z > \frac{1}{3})} = \frac{0.1587}{0.3707} \simeq \frac{1}{2}.$$

3. If x has a p.d.f. $f(x) = 1/3$ for $x = 1, 2, 3$ and $f(x) = 0$ elsewhere, find the p.d.f. of $y = 2x + 1$.

4. Let x have the p.d.f. $f(x) = 1; 0 \leq x \leq 1$. Find the p.d.f. of $y = -2 \log x$.

Answer: We have $y = y(x) = -2 \log x$ and $x = x(y) = e^{-y/2}$ Hence

$$\frac{dx}{dy} = -\frac{1}{2} e^{-\frac{1}{2}y} \quad \text{and, therefore,}$$

$$g(y) = f\{x(y)\} \left| \frac{dx}{dy} \right| = \frac{1}{2} e^{-\frac{1}{2}y}$$

is the pdf of y.

5. On average, 40% of those who reserve seats by telephone actually arrive at the theatre. A booking clerk has accepted six telephone reservations but has only four vacant seats. What is the probability that persons who have reserved seats will not be accommodated?

Answer: The probability that x out of 6 theatregoers will arrive is given by

$$b\Big(x; n = 6, p = \frac{4}{10}\Big) = \frac{6!}{(6 - x)!x!} \left(\frac{4}{10}\right)^x \left(\frac{6}{10}\right)^{6-x}.$$

The probability that persons will have to be turned away is

$$\begin{aligned} b(5) + b(6) &= 6 \left(\frac{4}{10}\right)^5 \left(\frac{6}{10}\right) + \left(\frac{4}{10}\right)^6 \\ &= \frac{4^5}{10^6}(36 + 4) = 0.040960. \end{aligned}$$

6. Equal numbers of men and women are represented on a list of 10 surnames. If the names are paired at random, what is the probability that each of the pairs will be of mixed sex?

Answer: There are 10! different sequences. Each of these represents a list of pairs wherein the order of the pairs is distinguished as well as the order of the partners within the pairs. There are 5! different orderings of the pairs and there are 2^5 ways in which the precedence of the individuals within the pair can be assigned. When these unessential distinctions are eliminated, we have

$$\frac{10!}{5!2^5} = 45 \times 21 = 945$$

distinct pairings. There are $5! = 120$ different mixed-sex pairs. Therefore the probability that all the pairs will be of mixed sex is

$$\frac{120}{945} \simeq \frac{1}{8}.$$