EXERCISES IN STATISTICS

Series A, No. 3

1. If $f(x) = a + bx^2; \quad 0 \leq x \leq 1$, determine a and b such that $E(x) = \frac{3}{4}$.

Answer: The condition that $\int f(x)dx = 1$, which is fulfilled by any probability density function, can be expressed as

$$\int_0^1 (a + bx^2)dx = \left[ax + \frac{bx^3}{3}\right]_0^1 = 1. \tag{1}$$

The condition that $E(x) = 3/4$ can be written as

$$\int_0^1 x(a + bx^2)dx = \left[\frac{ax^2}{2} + \frac{bx^4}{4}\right]_0^1 = \frac{3}{4}. \tag{2}$$

From (1) we have

$$a + \frac{b}{3} = 1 \quad \Longleftrightarrow \quad 3a + b = 3,$$

and, from (2), we have

$$\frac{a}{2} + \frac{b}{4} = \frac{3}{4} \quad \Longleftrightarrow \quad 2a + b = 3.$$

It follows that $a = 0$, $b = 3$.

2. Let $f(x) = 1; \quad 0 \leq x \leq 1$. Find

   (a) the mean and variance of x,

   (b) the mean and variance of x^2.

Answer:

$$E(x) = \int_0^1 x dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

$$E(x^2) = \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$$\begin{aligned} V(x) &= E(x^2) - \{E(x)\}^2 \\ &= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}. \end{aligned}$$

Next

$$E(x^4) = \int_0^1 x^4 dx = \left[\frac{x^5}{5}\right]_0^1 = \frac{1}{5},$$

$$V(x^2) = E(x^4) - \left\{E(x^2)\right\}^2$$
$$= \frac{1}{5} - \frac{1}{9} = \frac{4}{45}.$$

3. The probability that x buses will pass me before the 106 arrives is given by $f(x) = \frac{1}{4}\left(\frac{3}{4}\right)^x$. What is the probability that another five buses will pass me before the 106 arrives, given that three have already passed?

Answer: The buses which stop here are the 106 the 253 and the 277. We have $P(106) = \frac{1}{4}$ and $P(277 \cup 254) = 1 - P(106) = \frac{3}{4}$ as the probabilities for the next arrival. Moreover, these probabilities are independent. Therefore the probability that another five buses will pass before the 106 arrives is just

$$P(277 \cup 254)^5 P(106) = \frac{1}{4}\left(\frac{3}{4}\right)^5;$$

and this holds regardless of the fact that three buses which were not the 106 have already passed.

4. Six dice are tossed. What is the probability that every possible number will occur?

Answer: The number of possible outcomes from throwing the six dice is 6^6. The number of outcomes which are favourable to the event in question is the number of permutations of the integers $1, 2, \ldots, 6$ which is $6!$. Therefore

$$\text{Probability of the event} = \frac{\text{Number of outcomes favourable to the event}}{\text{Total number of outcomes}}$$
$$= \frac{6!}{6^6} = \frac{5 \times 4}{6^4} = 0.015432.$$

5. How many ways can six people sit down to a table laid for six? How many ways if the table is laid for eight? How many ways if the only concern is who sits next to whom?

Answer:

(a) Six people at a table laid for 6 can sit in $6! = 720$ different ways.

(b) If the table were laid for eight and there were eight people, then the number of ways would be 8! Imagine that there are six living persons and two ghosts: the ghost of Hamlets father and the ghost of Banquo. They can interchange their seating positions without the other persons noticing. Therefore there are only $8!/2 = 20160$ seating arrangements which are noticeably different.

(c) If the relative seating positions around a circular table are the only concern, then we may fix one person and permute the position of the remaining five to derive $5! = 240$ different arrangements. Alternatively, permute the 6 people and then divide the number of seating arrangements by 6 in recognition of the fact that we do not wish to distinguish amongst arrangements which are derived by rotating. Finally, if we are unconcerned about whether a person sits on the left or the right of another, then we must allow the order of the persons seated around the tabel to be reversed. In that case, we can divide by 2 to derive 120 different arrangements.

6. Ten balls are tossed into four boxes so that each ball is equally likely to fall into any box. What is the probability density function for the number of balls in the last box?

Answer: We have a binomial distribution

$$b\left(n=10, p=\tfrac{1}{4}\right) = \frac{10!}{(10-x)!x!}\left(\frac{1}{4}\right)^{x}\left(\frac{3}{4}\right)^{10-x}.$$

7. Ten percent of the words spoken by a politician are "Er". How many words must he speak so that the probability of at least one "Er"is 0.95?

Answer: On the assumption that the spoken word is a Bernoulli trial, the probability of at least one "Er"in n words is

$$1 - P(\text{no "Er"}) = 1 - (0.9)^n.$$

We can find the value of n by solving the equation $1-(0.9)^n = 0.95$:

$$n = \frac{\log 0.05}{\log 0.90} \simeq 29.$$

8. How many different ways are there of putting the spots on a die?

Answer: There are two sets of axes in 3-space: the clockwise and the anti-clockwise. We can imagine fixing the numbers to the vertices of these axes. Therefore the basic question is how many different ways are there of pairing 6 objects.

Let us begin with the permutation of 6 objects : 6! Each permutation implies a set of pairs if we take the ordered elements two-by-two.

Within each permutation the ordering of the pairs does not matter, so we must divide the number of permutation by 3!.

Nor does the order of precedence within the pairs matter, so we must divide again by 2^3.

The number which we derive is

$$\frac{6!}{3!2^3} = 15.$$

But, if we recall that there are two sets of axes, then we see that the total number of ways of fixing the spots on a die is 30.