STATISTICAL FOURIER ANALYSIS

The Fourier Representation of a Sequence

According to the basic result of Fourier analysis, it is always possible to approximate an arbitrary analytic function defined over a finite interval of the real line, to any desired degree of accuracy, by a weighted sum of sine and cosine functions of harmonically increasing frequencies. The accuracy of approximation increases with the number of functions within the sum.

Similar results apply in the case of sequences, which may be regarded as functions mapping from the set of integers onto the real line. For a sample of T observations $y_0, \ldots, y_{T-1}$, it is possible to devise an expression in the form

$$y_t = \sum_{j=0}^{n} \Big\{ \alpha_j \cos(\omega_j t) + \beta_j \sin(\omega_j t) \Big\}, \tag{1}$$

wherein $\omega_j = 2\pi j/T$ is a multiple of the fundamental frequency $\omega_1 = 2\pi/T$. Thus, the elements of a finite sequence can be expressed exactly in terms of sines and cosines. This expression is called the Fourier decomposition of y_t and the set of coefficients $\{\alpha_j, \beta_j; j = 0, 1, \ldots, n\}$ are called the Fourier coefficients.

When T is even, we have $n = T/2$; and it follows that

$$\begin{aligned} \sin(\omega_0 t) &= \sin(0) = 0, \\ \cos(\omega_0 t) &= \cos(0) = 1, \\ \sin(\omega_n t) &= \sin(\pi t) = 0, \\ \cos(\omega_n t) &= \cos(\pi t) = (-1)^t. \end{aligned} \tag{2}$$

Therefore, equation (1) becomes

$$y_t = \alpha_0 + \sum_{j=1}^{n-1} \Big\{ \alpha_j \cos(\omega_j t) + \beta_j \sin(\omega_j t) \Big\} + \alpha_n (-1)^t. \tag{3}$$

When T is odd, we have $n = (T-1)/2$; and then equation (1) becomes

$$y_t = \alpha_0 + \sum_{j=1}^{n} \Big\{ \alpha_j \cos(\omega_j t) + \beta_j \sin(\omega_j t) \Big\}. \tag{4}$$

In both cases, there are T nonzero coefficients amongst the set $\{\alpha_j, \beta_j; j = 0, 1, \ldots, n\}$; and the mapping from the sample values to the coefficients constitutes a one-to-one invertible transformation.

In equation (3), the frequencies of the trigonometric functions range from $\omega_1 = 2\pi/T$ to $\omega_n = \pi$; whereas, in equation (4), they range from $\omega_1 = 2\pi/T$ to $\omega_n = \pi(T-1)/T$. The frequency π is the so-called Nyquist frequency.

Although the process generating the data may contain components of frequencies higher than the Nyquist frequency, these will not be detected when it is sampled regularly at unit intervals of time. In fact, the effects on the process of components with frequencies in excess of the Nyquist value will be confounded with those whose frequencies fall below it.

To demonstrate this, consider the case where the process contains a component which is a pure cosine wave of unit amplitude and zero phase whose frequency ω lies in the interval $\pi < \omega < 2\pi$. Let $\omega^* = 2\pi - \omega$. Then

$$\begin{aligned} \cos(\omega t) &= \cos\big\{(2\pi - \omega^*)t\big\} \\ &= \cos(2\pi)\cos(\omega^* t) + \sin(2\pi)\sin(\omega^* t) \\ &= \cos(\omega^* t); \end{aligned} \tag{5}$$

which indicates that ω and ω^* are observationally indistinguishable. Here, $\omega^* < \pi$ is described as the alias of $\omega > \pi$.

The Spectral Representation of a Stationary Process

By allowing the value of n in the expression (1) to tend to infinity, it is possible to express a sequence of indefinite length in terms of a sum of sine and cosine functions. However, in the limit as $n \to \infty$, the coefficients α_j, β_j tend to vanish. Therefore, an alternative representation in terms of differentials is called for.

By writing $\alpha_j = dA(\omega_j)$, $\beta_j = dB(\omega_j)$, where $A(\omega)$, $B(\omega)$ are step functions with discontinuities at the points $\{\omega_j; j = 0, \ldots, n\}$, the expression (1) can be rendered as

$$y_t = \sum_j \Big\{\cos(\omega_j t) dA(\omega_j) + \sin(\omega_j t) dB(\omega_j)\Big\}. \tag{6}$$

In the limit, as $n \to \infty$, the summation is replaced by an integral to give

$$y(t) = \int_0^\pi \Big\{\cos(\omega t) dA(\omega) + \sin(\omega t) dB(\omega)\Big\}. \tag{7}$$

Here, $\cos(\omega t)$ and $\sin(\omega t)$, and therefore $y(t)$, may be regarded as infinite sequences defined over the entire set of positive and negative integers.

Since $A(\omega)$ and $B(\omega)$ are discontinuous functions for which no derivatives exist, one must avoid using $\alpha(\omega)d\omega$ and $\beta(\omega)d\omega$ in place of $dA(\omega)$ and $dB(\omega)$. Moreover, the integral in equation (7) is a Fourier–Stieltjes integral.

In order to derive a statistical theory for the process that generates $y(t)$, one must make some assumptions concerning the functions $A(\omega)$ and $B(\omega)$. So far, the sequence $y(t)$ has been interpreted as a realisation of a stochastic process. If $y(t)$ is regarded as the stochastic process itself, then the functions $A(\omega)$, $B(\omega)$ must, likewise, be regarded as stochastic processes defined over the interval $[0, \pi]$. A single realisation of these processes now corresponds to a single realisation of the process $y(t)$.

The first assumption to be made is that the functions $dA(\omega)$ and $dB(\omega)$ represent a pair of stochastic processes of zero mean which are indexed on the continuous parameter ω. Thus

$$E\big\{dA(\omega)\big\} = E\big\{dB(\omega)\big\} = 0. \tag{8}$$

The second and third assumptions are that the two processes are mutually uncorrelated and that non-overlapping increments within each process are uncorrelated. Thus

$$\begin{aligned} E\big\{dA(\omega)dB(\lambda)\big\} &= 0 \quad \text{for all} \quad \omega, \lambda, \\ E\big\{dA(\omega)dA(\lambda)\big\} &= 0 \quad \text{if} \quad \omega \neq \lambda, \\ E\big\{dB(\omega)dB(\lambda)\big\} &= 0 \quad \text{if} \quad \omega \neq \lambda. \end{aligned} \tag{9}$$

The final assumption is that the variance of the increments is given by

$$\begin{aligned} V\big\{dA(\omega)\big\} = V\big\{dB(\omega)\big\} &= 2dF(\omega) \\ &= 2f(\omega)d\omega. \end{aligned} \tag{10}$$

We can see that, unlike $A(\omega)$ and $B(\omega)$, $F(\omega)$ is a continuous differentiable function. The function $F(\omega)$ and its derivative $f(\omega)$ are the spectral distribution function and the spectral density function, respectively.

In order to express equation (7) in terms of complex exponentials, we may define a pair of conjugate complex stochastic processes:

$$\begin{aligned} dZ(\omega) &= \frac{1}{2}\big\{dA(\omega) - idB(\omega)\big\}, \\ dZ^*(\omega) &= \frac{1}{2}\big\{dA(\omega) + idB(\omega)\big\}. \end{aligned} \tag{11}$$

Also, we may extend the domain of the functions $A(\omega)$, $B(\omega)$ from $[0, \pi]$ to $[-\pi, \pi]$ by regarding $A(\omega)$ as an even function such that $A(-\omega) = A(\omega)$ and by regarding $B(\omega)$ as an odd function such that $B(-\omega) = -B(\omega)$. Then, there is

$$dZ^*(\omega) = dZ(-\omega). \tag{12}$$

From conditions under (9), it follows that

$$\begin{aligned} E\big\{dZ(\omega)dZ^*(\lambda)\big\} &= 0 \quad \text{if} \quad \omega \neq \lambda, \\ E\{dZ(\omega)dZ^*(\omega)\} &= f(\omega)d\omega. \end{aligned} \tag{13}$$

These results may be used to reexpress equation (7) as

$$
\begin{aligned}
y(t) &= \int_0^{\pi} \left\{ \frac{(e^{i\omega t} + e^{-i\omega t})}{2} dA(\omega) - i\frac{(e^{i\omega t} - e^{-i\omega t})}{2} dB(\omega) \right\} \\
&= \int_0^{\pi} \left\{ e^{i\omega t} \frac{\{dA(\omega) - idB(\omega)\}}{2} + e^{-i\omega t} \frac{\{dA(\omega) + idB(\omega)\}}{2} \right\} \qquad (14) \\
&= \int_0^{\pi} \left\{ e^{i\omega t} dZ(\omega) + e^{-i\omega t} dZ^*(\omega) \right\}.
\end{aligned}
$$

When the integral is extended over the range $[-\pi, \pi]$, this becomes

$$
y(t) = \int_{-\pi}^{\pi} e^{i\omega t} dZ(\omega). \qquad (15)
$$

This is commonly described as the spectral representation of the process $y(t)$.

The Autocovariances and the Spectral Density Function

The sequence of the autocovariances of the process $y(t)$ may be expressed in terms of the spectrum of the process. From equation (15), it follows that the autocovariance y_t at lag $\tau = t - k$ is given by

$$
\begin{aligned}
\gamma_\tau = C(y_t, y_k) &= E\left\{ \int_\omega e^{i\omega t} dZ(\omega) \int_\lambda e^{-i\lambda k} dZ(-\lambda) \right\} \\
&= \int_\omega \int_\lambda e^{i\omega t} e^{-i\lambda k} E\{dZ(\omega) dZ^*(\lambda)\} \qquad (16) \\
&= \int_\omega e^{i\omega\tau} E\{dZ(\omega) dZ^*(\omega)\} \\
&= \int_\omega e^{i\omega\tau} f(\omega) d\omega.
\end{aligned}
$$

Here the final equalities are derived by using the results (12) and (13). This equation indicates that the Fourier transform of the spectrum is the autocovariance function.

The inverse mapping from the autocovariances to the spectrum is given by

$$
\begin{aligned}
f(\omega) &= \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} \gamma_\tau e^{-i\omega\tau} \\
&= \frac{1}{2\pi} \left\{ \gamma_0 + 2 \sum_{\tau=1}^{\infty} \gamma_\tau \cos(\omega\tau) \right\}. \qquad (17)
\end{aligned}
$$

This function is directly comparable to the periodogram of a data sequence. However, the periodogram has T empirical autocovariances $c_0, \ldots, c_{T-1}$ in place of an indefinite number of theoretical autocovariances. Also, it differs from the spectrum by a scalar factor of 4π. In many texts, equation (17) serves as the primary definition of the spectrum.

To demonstrate the relationship which exists between equations (16) and (17), we may substitute the latter into the former to give

$$\begin{aligned}\gamma_\tau &= \int_{-\pi}^{\pi} e^{i\omega\tau}\Big\{\frac{1}{2\pi}\sum_{\tau=-\infty}^{\infty}\gamma_\tau e^{-i\omega\tau}\Big\}d\omega \\ &= \frac{1}{2\pi}\sum_{\kappa=-\infty}^{\infty}\gamma_\kappa\int_{-\pi}^{\pi} e^{i\omega(\tau-\kappa)}d\omega.\end{aligned} \tag{18}$$

From the fact that

$$\int_{-\pi}^{\pi} e^{i\omega(\tau-\kappa)}d\omega = \begin{cases} 2\pi, & \text{if } \kappa=\tau; \\ 0, & \text{if } \kappa\neq\tau, \end{cases} \tag{19}$$

it can be seen that the RHS of the equation reduces to γ_τ. This serves to show that equations (16) and (17) do indeed represent a Fourier transform and its inverse.

The essential interpretation of the spectral density function is indicated by the equation

$$\gamma_0 = \int_\omega f(\omega)d\omega, \tag{20}$$

which comes from setting $\tau = 0$ in equation (16). This equation shows how the variance or 'power' of $y(t)$, which is γ_0, is attributed to the cyclical components of which the process is composed.

It is easy to see that a flat spectrum corresponds to the autocovariance function which characterises a white-noise process $\varepsilon(t)$. Let $f_\varepsilon = f_\varepsilon(\omega)$ be the flat spectrum. Then, from equation (16), it follows that

$$\begin{aligned}\gamma_0 &= \int_{-\pi}^{\pi} f_\varepsilon(\omega)d\omega \\ &= 2\pi f_\varepsilon,\end{aligned} \tag{21}$$

and, from equation (16), it als0 follows that

$$\begin{aligned}\gamma_\tau &= \int_{-\pi}^{\pi} f_\varepsilon(\omega)e^{i\omega\tau}d\omega \\ &= f_\varepsilon\int_{-\pi}^{\pi} e^{i\omega\tau}d\omega \\ &= 0.\end{aligned} \tag{22}$$

These are the same as the conditions that serve to define a white-noise process. When the variance is denoted by σ^2_ε, the expression for the spectrum of the white-noise process becomes

$$f_\varepsilon(\omega) = \frac{\sigma^2_\varepsilon}{2\pi}. \tag{23}$$