# ALGEBRAIC POLYNOMIALS

Consider the equation $\phi_0+\phi_1 z+\phi_2 z^2=0$. Once the equation has been divided by $\phi_2$, it can be factorised as $(z-\lambda_1)(z-\lambda_2)$ where $\lambda_1$, $\lambda_2$ are the roots or zeros of the equation which are given by the formula

$$\lambda=\frac{-\phi_1\pm\sqrt{\phi_1^2-4\phi_2\phi_0}}{2\phi_2}. \tag{1}$$

If $\phi_1^2\geq 4\phi_2\phi_0$, then the roots $\lambda_1$, $\lambda_2$ are real. If $\phi_1^2=4\phi_2\phi_0$, then $\lambda_1=\lambda_2$. If $\phi_1^2<4\phi_2\phi_0$, then the roots are the conjugate complex numbers $\lambda=\alpha+i\beta$, $\lambda^*=\alpha-i\beta$, where $i=\sqrt{-1}$.

There are three alternative ways of representing the conjugate complex numbers $\lambda$ and $\lambda^*$ :

$$\begin{aligned}\lambda&=\alpha+i\beta=\rho(\cos\theta+i\sin\theta)=\rho e^{i\theta},\\ \lambda^*&=\alpha-i\beta=\rho(\cos\theta-i\sin\theta)=\rho e^{-i\theta},\end{aligned} \tag{2}$$

where

$$\rho=\sqrt{\alpha^2+\beta^2}\qquad\text{and}\qquad\theta=\tan^{-1}\left(\frac{\beta}{\alpha}\right). \tag{3}$$

These are called, respectively, the Cartesian form, the trigonometrical form and the exponential form.

The Cartesian and trigonometrical representations are understood by considering the Argand diagram:

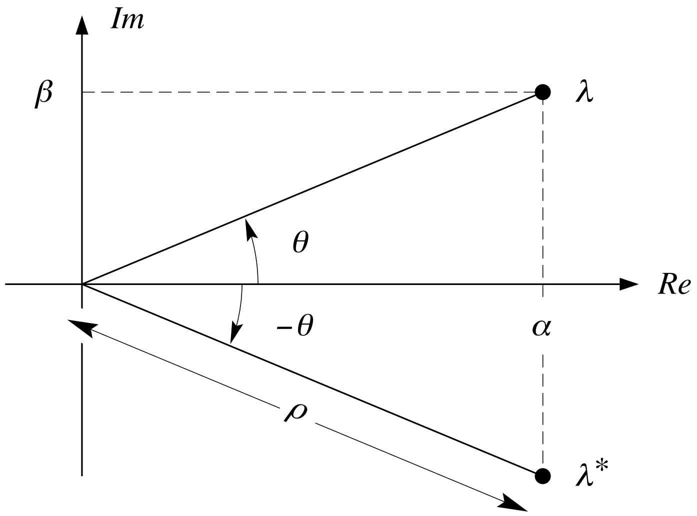


**Figure 1.** The Argand Diagram showing a complex number $\lambda=\alpha+i\beta$ and its conjugate $\lambda^*=\alpha-i\beta$.

The exponential form is understood by considering the following series expansions of $\cos\theta$ and $i\sin\theta$ about the point $\theta = 0$:

$$\begin{aligned} \cos\theta &= \left\{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right\}, \\ i\sin\theta &= \left\{i\theta - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} - \frac{i\theta^7}{7!} + \cdots\right\}. \end{aligned} \tag{4}$$

Adding these gives

$$\begin{aligned} \cos\theta + i\sin\theta &= \left\{1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots\right\} \\ &= e^{i\theta}. \end{aligned} \tag{5}$$

Likewise, by subtraction, we get

$$\begin{aligned} \cos\theta - i\sin\theta &= \left\{1 - i\theta - \frac{\theta^2}{2!} + \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} - \cdots\right\} \\ &= e^{-i\theta}. \end{aligned} \tag{6}$$

These are Euler's equations. It follows from adding (5) and (6) that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}. \tag{7}$$

Subtracting (6) from (5) gives

$$\begin{aligned} \sin\theta &= \frac{-i}{2}(e^{i\theta} - e^{-i\theta}) \\ &= \frac{1}{2i}(e^{i\theta} - e^{-i\theta}). \end{aligned} \tag{8}$$

Now consider the general equation of the $n$th order:

$$\phi_0 + \phi_1 z + \phi_2 z^2 + \cdots + \phi_n z^n = 0. \tag{9}$$

On dividing by $\phi_n$, we can factorise this as

$$(z - \lambda_1)(z - \lambda_2)\cdots(z - \lambda_n) = 0, \tag{10}$$

where some of the roots may be real and others may be complex. The complex roots come in conjugate pairs, so that, if $\lambda = \alpha + i\beta$ is a complex root, then there is a corresponding root $\lambda^* = \alpha - i\beta$ such that the product $(z-\lambda)(z-\lambda^*) = z^2 - 2\alpha z + (\alpha^2 + \beta^2)$ is real and quadratic. When we multiply the $n$ factors together, we obtain the expansion

$$0 = z^n - \sum_i \lambda_i z^{n-1} + \sum_i \sum_j \lambda_i \lambda_j z^{n-2} - \cdots (-1)^n \lambda_1 \lambda_2 \cdots \lambda_n. \tag{ii}$$

This can be compared with the expression $(\phi_0/\phi_n)+(\phi_1/\phi_n)z+\cdots+z^n = 0$. By equating coefficients of the two expressions, we find that $(\phi_0/\phi_n) = (-1)^n\prod\lambda_i$ or, equivalently,

$$\phi_n = \phi_0\prod_{i=1}^{n}(-\lambda_i)^{-1}. \tag{12}$$

Thus we can express the polynomial in any of the following forms:

$$\begin{aligned}\sum\phi_i z^i &= \phi_n\prod(z-\lambda_i)\\ &= \phi_0\prod(-\lambda_i)^{-1}\prod(z-\lambda_i)\\ &= \phi_0\prod\left(1-\frac{z}{\lambda_i}\right).\end{aligned} \tag{13}$$

We should also note that, if $\lambda$ is a root of the primary equation $\sum\phi_i z^i = 0$, where rising powers of $z$ are associated with rising indices on the coefficients, then $\mu = 1/\lambda$ is a root of the equation $\sum\phi_i z^{n-i} = 0$, which has declining powers of $z$ instead. This follows since $\sum\phi_i\lambda^i = \sum\phi_i\mu^{-i} = 0$ implies that $\mu^n\sum\phi_i\mu^{-i} = \sum\phi_i\mu^{n-i} = 0$. Confusion can arise from not knowing which of the two equations one is dealing with.

### Rational Functions of Polynomials

If $\beta(z)$ and $\phi(z)$ are polynomial functions of $z$ of degrees $d$ and $g$ respectively with $d < g$, then the ratio $\beta(z)/\phi(z)$ is described as a proper rational function. We shall often encounter expressions of the form

$$y(t) = \frac{\beta(L)}{\phi(L)}x(t). \tag{14}$$

For this to have a meaningful interpretation in the context of a time-series model, we normally require that $y(t)$ should be a bounded sequence whenever $x(t)$ is bounded. The necessary and sufficient condition for the boundedness of $y(t)$, in that case, is that the series expansion of $\beta(z)/\phi(z)$ should be convergent whenever $|z| \leq 1$. We can determine whether or not the sequence will converge by expressing the ratio $\beta(z)/\phi(z)$ as a sum of partial fractions. The basic result is as follows:

(15) If $\beta(z)/\phi(z) = \beta(z)/\{\phi_1(z)\phi_2(z)\}$ is a proper rational function, and if $\phi_1(z)$ and $\phi_2(z)$ have no common factor, then the function can be uniquely expressed as

$$\frac{\beta(z)}{\phi(z)} = \frac{\beta_1(z)}{\phi_1(z)} + \frac{\beta_2(z)}{\phi_2(z)},$$

where $\beta_1(z)/\phi_1(z)$ and $\beta_2(z)/\phi_2(z)$ are proper rational functions.

**Example.** Consider the following partial fraction expansion

$$\frac{gz+d}{(1-z/\lambda_1)(1-z/\lambda_2)}=\frac{c_1}{(1-z/\lambda_1)}+\frac{c_2}{(1-z/\lambda_2)} \tag{16}$$

To find the value of $c_1$, the equation is multiplied on both sides by $1-z/\lambda_1$ and then $z$ is set to $\lambda_1$. The result is that $c_1=(g\lambda_1+d)/(1-\lambda_1/\lambda_2)$. Likewise, it is found that $c_2=(g\lambda_2+d)/(1-\lambda_2/\lambda_1)$.

Imagine that $\phi(z)=\prod(1-z/\lambda_i)$. Then, repeated applications of this basic result enables us to write

$$\frac{\beta(z)}{\phi(z)}=\frac{\kappa_1}{1-z/\lambda_1}+\frac{\kappa_2}{1-z/\lambda_2}+\cdots+\frac{\kappa_g}{1-z/\lambda_g}. \tag{16}$$

By adding the terms on the RHS, we find an expression with a numerator of degree $n-1$. By equating the terms of the numerator with the terms of $\beta(z)$, we can find the values $\kappa_1,\ \kappa_2,\ldots,\kappa_g$. The convergence of the expansion of $\beta(z)/\phi(z)$ is a straightforward matter. For the series converges if and only if the expansion of each of the partial fractions converges. For the expansion

$$\frac{\kappa}{1-z/\lambda}=\kappa\Big\{1+z/\lambda+(z/\lambda)^2+\cdots\Big\} \tag{17}$$

to converge when $|z|\leq 1$, it is necessary and sufficient that $|\lambda|>1$.

The expansion of (17) applies to complex roots as well as to real roots:

$$\begin{aligned}\frac{c}{1-z/\lambda}+\frac{c^*}{1-z/\lambda^*}=&c\left\{1+z/\lambda+(z/\lambda)^2+\cdots\right\}\\ &+c^*\left\{1+z/\lambda^*+(z/\lambda^*)^2+\cdots\right\}\\ =&\sum_{t=0}^{\infty}z^t(c\lambda^{-t}+c^*\lambda^{*-t}).\end{aligned} \tag{18}$$

The various complex quantities can be represented in terms of exponentials:

$$\begin{aligned}\lambda&=\kappa^{-1}e^{-\mathrm{i}\omega}, & \lambda^*&=\kappa^{-1}e^{\mathrm{i}\omega},\\ &=\rho e^{-\mathrm{i}\theta}, & c^*&=\rho e^{\mathrm{i}\theta}.\end{aligned} \tag{19}$$

Then, the generic term of the expansion becomes

$$\begin{aligned}z^t(c\lambda^{-t}+c^*\lambda^{*-t})&=z^t\rho\kappa^t\big\{e^{\mathrm{i}(\omega t-\theta)}+e^{-\mathrm{i}(\omega t-\theta)}\big\}\\ &=z^t2\rho\kappa^t\cos(\omega t-\theta).\end{aligned} \tag{20}$$

The expansion converges for all $|z| \leq 1$ if and only if $|\kappa| < 1$, which is a condition on the modulus of the complex number $\lambda$. But, $|\kappa| = |\lambda^{-1}| = |\lambda|^{-1}$; so it is confirmed that the necessary and sufficient condition for convergence is that $|\lambda| > 1$.

Finally, consider the case of a repeated root with a multiplicity of $n$. Then, a Maclaurin series expansion is available that gives

$$(21)\qquad \frac{1}{(1 - z/\lambda)^n} = 1 + n\frac{z}{\lambda} + \frac{n(n+1)}{2!}\left(\frac{z}{\lambda}\right)^2 + \frac{n(n+1)(n+2)}{3!}\left(\frac{z}{\lambda}\right)^3 + \cdots.$$

If $\lambda$ is real, then $|\lambda| > 1$ is the condition for convergence. If $\lambda$ is complex, then it can be combined with the conjugate root in the manner of (20) to create a trigonometric function; and, again, the condition for convergence is that $|\lambda| > 1$.

This result can be understood by regarding the LHS of (21) as a representation of $n$ transfer functions in series, each of which fulfils the BIBO condition.

The general conclusion is that the transfer function is stable if and only if all of the roots of the denominator polynomial $a(z)$, which are described as the poles of the transfer function, lie outside the unit circle in the complex plane.

## The Expansion of a Rational Function: The Method of Detached Coefficients

The method of finding the coefficients of the series expansion of the transfer function in the general case can be illustrated by the second-order case:

$$(22)\qquad \frac{\beta_0 + \beta_1 z}{1 - \phi_1 z - \phi_2 z^2} = \{\omega_0 + \omega_1 z + \omega_2 z^2 + \cdots\}.$$

We rewrite this equation as

$$(23)\qquad \beta_0 + \beta_1 z = \{1 - \phi_1 z - \phi_2 z^2\}\{\omega_0 + \omega_1 z + \omega_2 z^2 + \cdots\}.$$

Then, by performing the multiplication on the RHS, and by equating the coefficients of the same powers of $z$ on the two sides of the equation, we find that

$$(24)\qquad \begin{aligned} \beta_0 &= \omega_0, & \omega_0 &= \beta_0,\\ \beta_1 &= \omega_1 - \phi_1\omega_0, & \omega_1 &= \beta_1 + \phi_1\omega_0,\\ 0 &= \omega_2 - \phi_1\omega_1 - \phi_2\omega_0, & \omega_2 &= \phi_1\omega_1 + \phi_2\omega_0,\\ &\vdots & &\vdots\\ 0 &= \omega_n - \phi_1\omega_{n-1} - \phi_2\omega_{n-2}, & \omega_n &= \phi_1\omega_{n-1} + \phi_2\omega_{n-2}. \end{aligned}$$