THE FOURIER DECOMPOSITION OF A TIME SERIES

In spite of the notion that a regular trigonometrical function is an inappropriate means for modelling an economic cycle other than a seasonal fluctuation, there are good reasons for explaining a data sequence in terms of such functions.

The Fourier decomposition of a series is a matter of explaining the series entirely as a composition of sinusoidal functions. Thus it is possible to represent the generic element of the sample as

$$y_t = \sum_{j=0}^{n} \Big\{ \alpha_j \cos(\omega_j t) + \beta_j \sin(\omega_j t) \Big\}. \tag{1}$$

Assuming that $T = 2n$ is even, this sum comprises T functions whose frequencies

$$\omega_j = \frac{2\pi j}{T}, \quad j = 0, \ldots, n = \frac{T}{2} \tag{2}$$

are at equally spaced points in the interval $[0, \pi]$.

As we might infer from our analysis of a seasonal fluctuation, there are as many nonzeros elements in the sum under (1) as there are data points, for the reason that two of the functions within the sum—namely $\sin(\omega_0 t) = \sin(0)$ and $\sin(\omega_n t) = \sin(\pi t)$—are identically zero. It follows that the mapping from the sample values to the coefficients constitutes a one-to-one invertible transformation. The same conclusion arises in the slightly more complicated case where T is odd.

The angular velocity $\omega_j = 2\pi j/T$ relates to a pair of trigonometrical components which accomplish j cycles in the T periods spanned by the data. The highest velocity $\omega_n = \pi$ corresponds to the so-called Nyquist frequency. If a component with a frequency in excess of π were included in the sum in (1), then its effect would be indistinguishable from that of a component with a frequency in the range $[0, \pi]$

To demonstrate this, consider the case of a pure cosine wave of unit amplitude and zero phase whose frequency ω lies in the interval $\pi < \omega < 2\pi$. Let $\omega^* = 2\pi - \omega$. Then

$$\begin{aligned} \cos(\omega t) &= \cos\big\{(2\pi - \omega^*)t\big\} \\ &= \cos(2\pi)\cos(\omega^* t) + \sin(2\pi)\sin(\omega^* t) \\ &= \cos(\omega^* t); \end{aligned} \tag{3}$$

which indicates that ω and ω^* are observationally indistinguishable. Here, $\omega^* \in [0, \pi]$ is described as the alias of $\omega > \pi$.

For an illustration of the problem of aliasing, let us imagine that a person observes the sea level at 6am. and 6pm. each day. He should notice a very gradual recession and advance of the water level; the frequency of the cycle being $f = 1/28$ which amounts to one tide in 14 days. In fact, the true frequency is $f = 1 - 1/28$ which gives 27 tides in 14 days. Observing the sea level every six hours should enable him to infer the correct frequency.

Calculation of the Fourier Coefficients

For heuristic purposes, we can imagine calculating the Fourier coefficients using an ordinary regression procedure to fit equation (1) to the data. In this case, there would be no regression residuals, for the reason that we are 'estimating' a total of T coefficients from T data points; so we are actually solving a set of T linear equations in T unknowns.

A reason for not using a multiple regression procedure is that, in this case, the vectors of 'explanatory' variables are mutually orthogonal. Therefore, T applications of a univariate regression procedure would be appropriate to our purpose.

Let $c_j = [c_{0j}, \ldots, c_{T-1,j}]'$ and $s_j = [s_{0,j}, \ldots, s_{T-1,j}]'$ represent vectors of T values of the generic functions $\cos(\omega_j t)$ and $\sin(\omega_j t)$ respectively. Then there are the following orthogonality conditions:

$$\begin{aligned} c_i'c_j &= 0 \quad \text{if} \quad i \neq j, \\ s_i's_j &= 0 \quad \text{if} \quad i \neq j, \\ c_i's_j &= 0 \quad \text{for all} \quad i, j. \end{aligned} \tag{4}$$

In addition, there are the following sums of squares:

$$\begin{aligned} c_0'c_0 &= c_n'c_n = T, \\ s_0's_0 &= s_n's_n = 0, \\ c_j'c_j &= s_j's_j = \frac{T}{2}. \end{aligned} \tag{5}$$

The 'regression' formulae for the Fourier coefficients are therefore

$$\alpha_0 = (i'i)^{-1}i'y = \frac{1}{T}\sum_t y_t = \bar{y}, \tag{6}$$

$$\alpha_j = (c_j'c_j)^{-1}c_j'y = \frac{2}{T}\sum_t y_t \cos\omega_i t, \tag{7}$$

$$\beta_j = (s_j's_j)^{-1}s_j'y = \frac{2}{T}\sum_t y_t \sin\omega_j t. \tag{8}$$

By pursuing the analogy of multiple regression, we can understand that there is a complete decomposition of the sum of squares of the elements of y which is given by

$$y'y = \alpha_0^2 i'i + \sum_j \alpha_j^2 c_j' c_j + \sum_j \beta_j^2 s_j' s_j. \tag{9}$$

Now consider writing $\alpha_0^2 i'i = \bar{y}^2 i'i = \bar{y}'\bar{y}$ where $\bar{y}' = [\bar{y}, \ldots, \bar{y}]$ is the vector whose repeated element is the sample mean $\bar{y}$. It follows that $y'y - \alpha_0^2 i'i = y'y - \bar{y}'\bar{y} = (y - \bar{y})'(y - \bar{y})$. Therefore, we can rewrite the equation as

$$(y - \bar{y})'(y - \bar{y}) = \frac{T}{2} \sum_j \left\{ \alpha_j^2 + \beta_j^2 \right\} = \frac{T}{2} \sum_j \rho_j^2, \tag{10}$$

and it follows that we can express the variance of the sample as

$$\begin{aligned} \frac{1}{T} \sum_{t=0}^{T-1} (y_t - \bar{y})^2 &= \frac{1}{2} \sum_{j=1}^{n} (\alpha_j^2 + \beta_j^2) \\ &= \frac{2}{T^2} \sum_j \left\{ \left(\sum_t y_t \cos \omega_j t \right)^2 + \left(\sum_t y_t \sin \omega_j t \right)^2 \right\}. \end{aligned} \tag{11}$$

The proportion of the variance which is attributable to the component at frequency ω_j is $(\alpha_j^2 + \beta_j^2)/2 = \rho_j^2/2$, where ρ_j is the amplitude of the component.

The number of the Fourier frequencies increases at the same rate as the sample size T. Therefore, if the variance of the sample remains finite, and if there are no regular harmonic components in the process generating the data, then we can expect the proportion of the variance attributed to the individual frequencies to decline as the sample size increases. If there is such a regular component within the process, then we can expect the proportion of the variance attributable to it to converge to a finite value as the sample size increases.

In order provide a graphical representation of the decomposition of the sample variance, we must scale the elements of equation (11) by a factor of T. The graph of the function $I(\omega_j) = (T/2)(\alpha_j^2 + \beta_j^2)$ is know as the periodogram.

There are many impressive examples where the estimation of the periodogram has revealed the presence of regular harmonic components in a data series which might otherwise have passed undetected. One of the best-know examples concerns the analysis of the brightness or magnitude of the star T. Ursa Major. It was shown by Whittaker and Robinson in 1924 that this series could be described almost completely in terms of two trigonometrical functions with periods of 24 and 29 days.

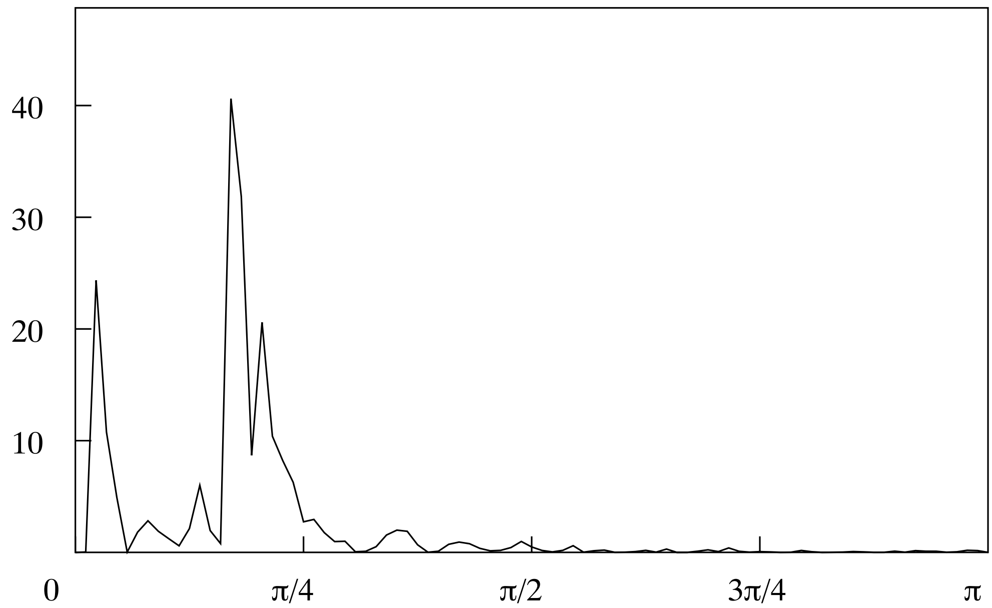

Figure 3. The periodogram of Wolfer's Sunspot Numbers 1749–1924.

The attempts to discover underlying components in economic time-series have been less successful. One application of periodogram analysis which was a notorious failure was its use by William Beveridge in 1921 and 1923 to analyse a long series of European wheat prices. The periodogram had so many peaks that at least twenty possible hidden periodicities could be picked out, and this seemed to be many more than could be accounted for by plausible explanations within the realms of economic history.

Such findings seem to diminish the importance of periodogram analysis in econometrics. However, the fundamental importance of the periodogram is established once it is recognised that it represents nothing less than the Fourier transform of the sequence of empirical autocovariances.

The Empirical Autocovariances

A natural way of representing the serial dependence of the elements of a data sequence is to estimate their autocovariances. The empirical autocovariance of lag τ is defined by the formula

$$c_\tau = \frac{1}{T}\sum_{t=\tau}^{T-1}(y_t - \bar{y})(y_{t-\tau} - \bar{y}). \tag{12}$$

The empirical autocorrelation of lag τ is defined by $r_\tau = c_\tau / c_0$ where c_0, which is formally the autocovariance of lag 0, is the variance of the sequence. The autocorrelation provides a measure of the relatedness of data points separated by τ periods which is independent of the units of measurement.

It is straightforward to establish the relationship between the periodogram and the sequence of autocovariances.

The periodogram may be written as

$$(13) \qquad I(\omega_j) = \frac{2}{T}\left[\left\{\sum_{t=0}^{T-1}\cos(\omega_j t)(y_t - \bar{y})\right\}^2 + \left\{\sum_{t=0}^{T-1}\sin(\omega_j t)(y_t - \bar{y})\right\}^2\right].$$

The identity $\sum_t \cos(\omega_j t)(y_t - \bar{y}) = \sum_t \cos(\omega_j t) y_t$ follows from the fact that, by construction, $\sum_t \cos(\omega_j t) = 0$ for all j. Expanding the expression in (38) gives

$$(14) \qquad \begin{aligned} I(\omega_j) =& \frac{2}{T}\left\{\sum_t\sum_s \cos(\omega_j t)\cos(\omega_j s)(y_t - \bar{y})(y_s - \bar{y})\right\} \\ &+ \frac{2}{T}\left\{\sum_t\sum_s \sin(\omega_j t)\sin(\omega_j s)(y_t - \bar{y})(y_s - \bar{y})\right\}, \end{aligned}$$

and, by using the identity $\cos(A)\cos(B) + \sin(A)\sin(B) = \cos(A - B)$, we can rewrite this as

$$(15) \qquad I(\omega_j) = \frac{2}{T}\left\{\sum_t\sum_s \cos(\omega_j[t - s])(y_t - \bar{y})(y_s - \bar{y})\right\}.$$

Next, on defining $\tau = t - s$ and writing $c_\tau = \sum_t (y_t - \bar{y})(y_{t-\tau} - \bar{y})/T$, we can reduce the latter expression to

$$(16) \qquad I(\omega_j) = 2\sum_{\tau=1-T}^{T-1}\cos(\omega_j \tau)c_\tau,$$

which is a Fourier transform of the sequence of empirical autocovariances.

An Appendix on Harmonic Cycles

Lemma 1. Let $\omega_j = 2\pi j/T$ where $j \in \{0, 1, \ldots, T/2\}$ if T is even and $j \in \{0, 1, \ldots, (T-1)/2\}$ if T is odd. Then

$$\sum_{t=0}^{T-1}\cos(\omega_j t) = \sum_{t=0}^{T-1}\sin(\omega_j t) = 0.$$

Proof. By Euler's equations, we have

$$\sum_{t=0}^{T-1}\cos(\omega_j t) = \frac{1}{2}\sum_{t=0}^{T-1}\exp(i2\pi jt/T) + \frac{1}{2}\sum_{t=0}^{T-1}\exp(-i2\pi jt/T).$$

By using the formula $1 + \lambda + \cdots + \lambda^{T-1} = (1 - \lambda^T)/(1 - \lambda)$, we find that

$$\sum_{t=0}^{T-1} \exp(i2\pi jt/T) = \frac{1 - \exp(i2\pi j)}{1 - \exp(i2\pi j/T)}.$$

But $\exp(i2\pi j) = \cos(2\pi j) + i\sin(2\pi j) = 1$, so the numerator in the expression above is zero, and hence $\sum_t \exp(i2\pi j/T) = 0$. By similar means, we can show that $\sum_t \exp(-i2\pi j/T) = 0$; and, therefore, it follows that $\sum_t \cos(\omega_j t) = 0$. An analogous proof shows that $\sum_t \sin(\omega_j t) = 0$.

Lemma 2. Let $\omega_j = 2\pi j/T$ where $j \in 0, 1, \ldots, T/2$ if T is even and $j \in 0, 1, \ldots, (T-1)/2$ if T is odd. Then

$$(a) \qquad \sum_{t=0}^{T-1} \cos(\omega_j t)\cos(\omega_k t) = \begin{cases} 0, & \text{if } j \neq k; \\ \frac{T}{2}, & \text{if } j = k. \end{cases}$$

$$(b) \qquad \sum_{t=0}^{T-1} \sin(\omega_j t)\sin(\omega_k t) = \begin{cases} 0, & \text{if } j \neq k; \\ \frac{T}{2}, & \text{if } j = k. \end{cases}$$

$$(c) \qquad \sum_{t=0}^{T-1} \cos(\omega_j t)\sin(\psi_k t) = 0 \qquad \text{if} j \neq k.$$

Proof. From the formula $\cos A \cos B = \frac{1}{2}\{\cos(A+B) + \cos(A-B)\}$ we have

$$\begin{aligned} \sum_{t=0}^{T-1} \cos(\omega_j t)\cos(\omega_k t) &= \frac{1}{2}\sum \{\cos([\omega_j + \omega_k]t) + \cos([\omega_j - \psi_k]t)\} \\ &= \frac{1}{2}\sum_{t=0}^{T-1} \{\cos(2\pi[j+k]t/T) + \cos(2\pi[j-k]t/T)\}. \end{aligned}$$

We find, in consequence of Lemma 1, that if $j \neq k$, then both terms on the RHS vanish, and thus we have the first part of (a). If $j = k$, then $\cos(2\pi[j-k]t/T) = \cos 0 = 1$ and so, whilst the first term vanishes, the second terms yields the value of T under summation. This gives the second part of (a).

The proofs of (b) and (c) follow along similar lines.

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