Rules of Differentiation

The Product Rule. If $u(x)$ and $v(x)$ are functions, continuous in an interval $[a, b]$ with derivatives $u'(a)$ and $v'(a)$ respectively at the point $x = a$, then the derivative of their product $p(x) = u(x)v(x)$ at that point is

$$p'(a) = u(a)v'(a) + v(a)u'(a).$$

Proof. Take a point $x > a$ which lies in (a, b). Then we may write

$$\begin{aligned}\frac{p(x) - p(a)}{x - a} &= \frac{u(x)v(x) - u(a)v(a)}{x - a} \\ &= \frac{u(x)\{v(x) - v(a)\} + v(a)\{u(x) - u(a)\}}{x - a} \\ &= u(x)\frac{v(x) - v(a)}{x - a} + v(a)\frac{u(x) - u(a)}{x - a}.\end{aligned}$$

Here $\lim(x \to a)v(x) = v(a)$. Also, by assumption, the limits as $x \to a$ of the two fractional parts in the line above exist and are equal to $v'(a)$ and $u'(a)$ respectively. Therefore

$$\lim_{x \to a} \frac{p(x) - p(a)}{x - a} = p'(a) = u(a)v'(a) + v(a)u'(a). \qquad \diamond$$

The schoolbook method of proving this result is to consider $y = uv$ and to suppose that, when x has a small increment δx, then u has the increment δu and v has the increment δv. Then

$$\begin{aligned}y + \delta y &= (u + \delta u)(v + \delta v) \\ &= uv + u\delta v + v\delta u + \delta u\delta v.\end{aligned}$$

Subtracting $y = uv$ from both sides and dividing by δx gives

$$\frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \frac{\delta u}{\delta x}\delta v.$$

Then it is argued that the ratios of the differentials tend towards the corresponding derivatives as $\delta x \to 0$ and that the final term disappears because $\delta v \to 0$. It should be noted that a proof such as this, which makes no reference to the values of the functions u and v at any specific point, presupposes the existence of the derivatives at all points.

Example. The so-called nominal value $V = pq$ of a manufacturing process is the product of the number of items q produced per unit period, ie. the quantity, and the unit price p. Both price and quantity are liable to change over time, leading to a change in the value of the process. Assuming that $p = p(t)$ and $q = q(t)$ are continuous differentiable functions of time, we have

$$\frac{dV}{dt} = p\frac{dq}{dt} + q\frac{dp}{dt}.$$

The proportional or percentage rate of change of the value is defined by

$$\begin{aligned}\frac{1}{V}\frac{dV}{dt} &= \frac{p}{V}\frac{dq}{dt} + \frac{q}{V}\frac{dp}{dt}\\ &= \frac{1}{q}\frac{dq}{dt} + \frac{1}{p}\frac{dp}{dt},\end{aligned}$$

which is the sum of the proportional changes in price and in quantity. If prices are increasing at a rate of 10% per annum and the quantity manufactured is growing at a rate of 15% per annum, then the nominal value of the output is changing at a rate of 25% per annum. These are instantaneous rates of change.

Imagine that price were to change over a twelve-month period by 10% and that quantity were to change by 15%. Denote prices at the start of the period by p_0 and at the end of the period by p_1. Use q_0 and q_1 likewise for quantity. Then

$$p_1 = 1.10 \times p_0 \qquad \text{and} \qquad q_1 = 1.15 \times q_0;$$

and the percentage change in the value of output over the period would be

$$\begin{aligned}\frac{V_1 - V_0}{V_0} &= \frac{p_1q_1 - p_0q_0}{p_0q_0}\\ &= 1.10 \times 1.15 - 1.0 = 0.265.\end{aligned}$$

That is to say, there is a $26\frac{1}{2}$% increase in value where one might have expected a 25% increase. This appears to contradict our previous finding. The seeming paradox is due to the fact that we are no longer dealing with instantaneous rates of change.

A little algebra may elucidate the matter. Let $\Delta p = p_1 - p_0$, $\Delta q = q_1 - q_0$ and $\Delta V = V_1 - V_0$, Then

$$\begin{aligned}V_1 &= (p_0 + \Delta p)(q_0 + \Delta q)\\ &= p_0q_0 + p_0\Delta q + q_0\Delta p + \Delta p\Delta q.\end{aligned}$$

and, therefore, the proportional change in value is

$$\begin{aligned}\frac{\Delta V}{V_0} &= \frac{p_0\Delta q + q_0\Delta p + \Delta p\Delta q}{p_0 q_0}\\ &= \frac{\Delta p}{p_0} + \frac{\Delta q}{q_0} + \frac{\Delta p\Delta q}{p_0 q_0}.\end{aligned}$$

In terms of our example, this equation reads

$$26\tfrac{1}{2}\% = 10\% + 15\% + \big\{10\% \times 15\%\big\}.$$

The Quotient Rule. If $u(x)$ and $v(x)$ are functions, continuous in an interval $[a, b]$ with derivatives $u'(a)$ and $v'(a)$, respectively, at the point $x = a$, then the derivative of their quotient $q(x) = u(v)/v(x)$ at that point is

$$q'(a) = \frac{v(a)u'(a) - u(a)v'(a)}{v^2(a)}.$$

Proof. Take a point $x > a$. Then we may write

$$\begin{aligned}\frac{q(x) - q(a)}{x - a} &= \frac{u(x)v(a) - v(x)u(a)}{(x - a)v(x)v(a)}\\ &= \frac{v(a)\{u(x) - u(a)\} + u(a)\{v(x) - v(a)\}}{(x - a)v(x)v(a)}\\ &= \frac{1}{v(x)v(a)}\left\{v(a)\frac{u(x) - v(a)}{x - a} - u(a)\frac{v(x) - v(a)}{x - a}\right\}.\end{aligned}$$

Here $v(a) \neq 0$ by assumption and, since $v(x)$ is continuous, there must be a neighbourhood of a in which $v(x) \neq 0$ also. Therefore there is no problem here of a "division by zero". Also the limits of the fractional parts of the expression above exist and are equal to $u'(a)$ and $v'(a)$ respectively. Therefore taking limits in the expression proves the theorem. ◇

One might wish to rephrase the proof in terms of differentials. Therefore let $y = u/v$ and consider

$$y + \delta y = \frac{u + \delta u}{v + \delta v}.$$

Subtracting $y = u/v$ from both sides gives

$$\begin{aligned}\delta y &= \frac{u + \delta u}{v + \delta v} - \frac{u}{v}\\ &= \frac{v(u + \delta u) - u(v + \delta v)}{v(v + \delta v)}.\end{aligned}$$

Dividing by δx gives

$$\frac{\delta y}{\delta x} = \frac{1}{v(v+\delta v)}\left\{v\left(\frac{u+\delta u}{\delta x}\right) + u\left(\frac{v+\delta v}{\delta x}\right)\right\}.$$

Then taking limits gives the derivative:

$$\frac{dy}{dx} = \frac{1}{v^2}\left\{v\frac{du}{dx} - u\frac{dv}{dx}\right\}.$$

Example. Let Y denote the gross national product (GNP) of a country and let N denote its population. Then $y = Y/N$ is the income per head. The quotient rule indicates that

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{Y}{N}\right) = \frac{1}{N^2}\left\{N\frac{dY}{dt} - Y\frac{dN}{dt}\right\}.$$

The proportional rate of growth of income per head is

$$\begin{aligned}\frac{1}{y}\frac{dy}{dt} &= \frac{1}{NY}\left\{N\frac{dy}{dt} - Y\frac{dN}{dt}\right\}\\ &= \frac{1}{Y}\frac{dY}{dt} - \frac{1}{N}\frac{dN}{dt}.\end{aligned}$$

Thus the growth in per capita income is evaluated by subtracting the population growth rate from the growth rate of GNP.