CONSTRAINED OPTIMISATION OF BIVARIATE FUNCTIONS

Constrained Optimisation of a Bivariate Function

It has been said that one of the defining problems of economics is that of the optimisation of functions subject to constraints. Examples are provided both by the theory of consumer demand and by the theory of the firm.

The consumers who are depicted by microeconomic theory are supposed to be maximising their personal utilities by choosing freely from amongst the available goods and services while facing the constraint of their limited budgets.

A manufacturer is depicted as maximising his profits subject to the constraints of the available technology which enables him to transform his raw materials into finished products. He may face other constraints as well, such as the shortage of capital funds or his inability, in the short run, to recruit skilled workers.

The techniques of constrained optimisation are best introduced by concentrating on a specific example from which the general principles may be extracted. The technique which is applied to many problems of constrained optimisation is that of Lagrangean multipliers which may appear at first to be of a somewhat arbitrary nature.

The problem which we shall select for our example is that of a consumer endeavouring to maximise his personal utility by distributing his budget optimally between two goods. We shall begin by solving the problem without the benefit of the Lagrangean technique.

The Consumer's Choice

Imagine that a consumer is endowed with a total budget of £M which is to be divided between two goods. If the prices of the two goods are denoted by p_1 and p_2 and if the quantities which are purchased are denoted by x_1 and x_2, then the condition that the budget is spent entirely is expressed by the equation

$$M = p_1x_1 + p_2x_2. \tag{1}$$

The graph of this equation represents a straight line which cuts the horizontal and vertical axes at the points $x_1 = M/p_1$ and $x_2 = M/p_2$ respectively. These are the quantities which could be purchased if the entire budget were devoted to the goods in question.

The slope of this budget line is given by $-p_1/p_2$. This result can be deduced in various ways. One way is to consider the movement along the line from the point of intersection with the horizontal axis to the point of intersection with the vertical axis. The ratio of the vertical and horizontal changes is

$$\frac{\Delta x_2}{\Delta x_1} = \left\{ \frac{M}{p_2} \Big/ - \frac{M}{p_1} \right\} = -\frac{p_1}{p_2}, \tag{2}$$

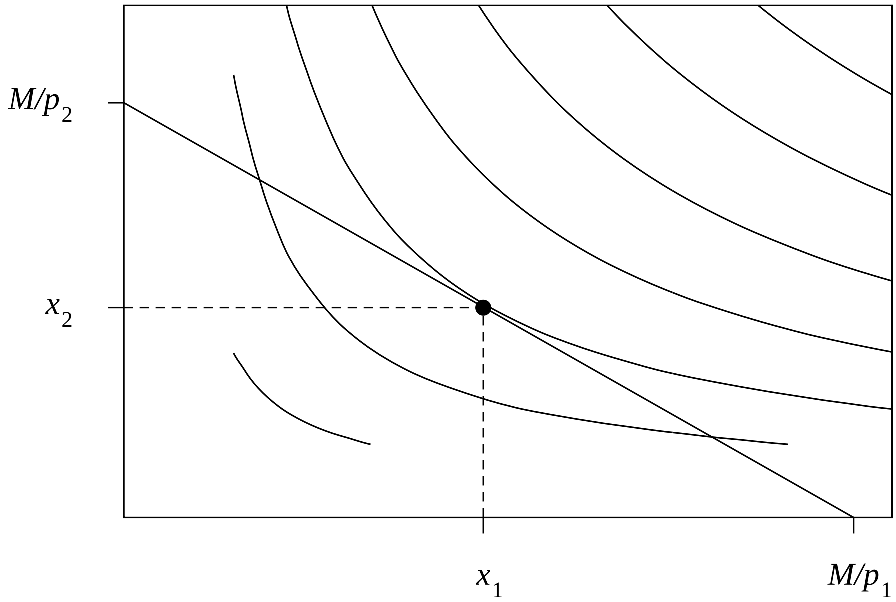

Figure 1. Utility is maximised at the point where the budget line is tangential to the highest attainable indifference curve.

which is the slope of the line. Equally, we may take differentials within the budget equation. This gives

$$dM = 0 = p_1 dx_1 + p_2 dx_2. \tag{3}$$

Here the condition $dM = 0$ reflects the fact that the budget is a fixed sum. By forming the ratio of dx_2 and dx_1, we obtain, once more, the slope of the budget line.

The consumer is supposed to choose how much of either good to purchase by assessing the overall utility which the goods generate in combination. The utility function may be denoted by $U = U(x_1, x_2)$. This is a function of the two quantities alone; and it is assumed that it is an increasing function of x_1 and of x_2. A further crucial assumption is made concerning the contours of the utility function which are the indifference curves in other words. These are assumed to be convex when viewed from the origin. We shall be able to justify this assumption shortly.

Each indifference curve represent a set of combinations of the goods—that is to say a set of points (x_1, x_2)—which afford equal utility to the consumer. Thus, if (dx_1, dx_2) is a small change from one point on the indifference curve to another point on the curve which is virtually contiguous, then it follows that

$$dU = 0 = \left(\frac{\partial U}{\partial x_1}\right) dx_1 + \left(\frac{\partial U}{\partial x_2}\right) dx_2. \tag{4}$$

Therefore the slope of the indifference curve is given by the formula

$$\frac{dx_2}{dx_1} = -\left\{ \frac{\partial U}{\partial x_1} \bigg/ \frac{\partial U}{\partial x_2} \right\}. \tag{5}$$

Given that the contours of the indifference curve are convex to the origin and given the assumption that U is an increasing function of x_1 and x_2, it now follows, from a simple geometric argument, that the highest accessible indifference curve is attained at a point where the slope of the budget line and that of the indifference curve coincide. Therefore the utility of the consumer is maximised when

$$\left\{ \frac{\partial U}{\partial x_1} \bigg/ \frac{\partial U}{\partial x_2} \right\} = \frac{p_1}{p_2}. \tag{6}$$

There are various ways in which this result may be expressed in order to heighten its intuitive appeal. One way is to declare that, at the optimal point, the ratio of the marginal utilities of the two goods—which is the LHS of the equation—must equal that rate at which the market allows the goods to be traded one for another—which is the price ratio of the RHS. An alternative formulation of the condition entails defining the quantities

$$\lambda_1 = \frac{1}{p_1}\frac{\partial U}{\partial x_1} \quad \text{and} \quad \lambda_2 = \frac{1}{p_2}\frac{\partial U}{\partial x_2}. \tag{7}$$

These are the satisfactions derived from marginal increases in the expenditures on either good. The condition of utility maximisation is that these two quantities should be equal. Clearly, if they were unequal, then extra utility could be derived by reducing the expenditure on one of the goods in order to increase the expenditure on the other.

Example. If the object of the exercise is to determine the quantities x_1 and x_2 of the two goods that are purchased by the consumer, then we should need to have an explicit expression for the utility function. Let us take, for example, the function

$$U(x_1, x_2) = \alpha x_1^{\beta} x_2^{\gamma}. \tag{8}$$

Then the two partial derivatives which are entailed by the condition of optimality are

$$\frac{\partial U}{\partial x_1} = \alpha\beta x_1^{\beta-1} x_2^{\gamma} \qquad \text{and} \qquad \frac{\partial U}{\partial x_2} = \alpha\gamma x_1^{\beta} x_2^{\gamma-1}. \tag{9}$$

Therefore the condition of optimality under (6) may be evaluated as

$$\frac{\beta x_2}{\gamma x_1} = \frac{p_2}{p_1} \qquad \text{or, equivalently,} \qquad \beta p_1 x_2 - \gamma p_2 x_1 = 0. \tag{10}$$

This is a simple linear equation in x_1 and x_2; and it may be solved in the company of the linear equation of the budget constraint given under (1) to find the two quantities.

The Lagrangean Method Constrained Optimisation

The method of Lagrangean multipliers entails converting the constrained optimisation problem into one which is seemingly unconstrained. In our illustrative bivariate optimisation problem, this could be achieved by recognising that, in effect, the consumer has only one degree of freedom which corresponds to the choice of where to locate the point (x_1, x_2) on the budget line. By eliminating one or other of the two variables—by setting $x_2 = M - p_1x_1/p_2$, or by setting $x_1 = M - p_2x_2/p_1$—an unconstrained univariate problem could be derived. Paradoxically, the method of Lagrange entails the introduction of a third variable which is the so-called Lagrangean multiplier λ.

The Lagrangean criterion function, which is an amended version of the utility function, takes the form of

$$L(x_1, x_2, \lambda) = U(x_1, x_2) + \lambda(M - p_1x_1 - p_2x_2), \tag{11}$$

Differentiating the function in respect of all three variables and setting the results to zero gives rise to the following first-order conditions:

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= \frac{\partial U}{\partial x_1} - \lambda p_1 = 0, \\ \frac{\partial L}{\partial x_2} &= \frac{\partial U}{\partial x_2} - \lambda p_2 = 0, \\ \frac{\partial L}{\partial \lambda} &= M - p_1x_1 - p_2x_2 = 0. \end{aligned} \tag{12}$$

The last of these is nothing but the equation (1) of the budget line. The first and the second equation can be rearranged to give

$$\frac{1}{p_1}\frac{\partial U}{\partial x_1} = \lambda \qquad \text{and} \qquad \frac{1}{p_2}\frac{\partial U}{\partial x_2} = \lambda. \tag{13}$$

These are exactly the equations which arise from (7) when the condition $\lambda_1 = \lambda_2 = \lambda_2$ is imposed. It may be recalled that this is the condition that the satisfaction derived from additional marginal expenditures is the same for both goods.

It should be be clear that the results obtained via the method of Lagrangean multipliers coincides exactly with those which we have obtained previously.