

EXPONENTIAL AND LOGISTIC GROWTH

Constant Proportional Growth

There is a close connection between the processes of geometric growth and exponential growth. Geometric growth is defined by the equation

$$(1) \quad y_t = y_0(1 + r)^t.$$

wherein y_0 stands for the value of the process at time 0 and y_t stands for the value at time t . Here t , which denotes time, is usually given only integer values; and the process is described as a discrete-time process. Geometric growth is constant proportional growth, which is to say the value of y increases by the same percentage in each time period. To see this, consider

$$(2) \quad \begin{aligned} \frac{\Delta y_t}{y_t} &= \frac{y_{t+1} - y_t}{y_t} \\ &= \frac{y_t(1 + r) - y_t}{y_t} = r. \end{aligned}$$

The same characteristic of constant proportional growth is manifested by an exponential process which is defined by

$$(3) \quad y_t = y_0 e^{\rho t}.$$

It is straightforward to confirm that

$$(4) \quad \begin{aligned} \frac{1}{y} \frac{dy}{dt} &= \frac{1}{y_0 e^{\rho t}} \times y_0 e^{\rho t} \rho \\ &= \rho. \end{aligned}$$

On comparing equations (1) and (3), we find that $y_t/y_0 = (1 + r)^t = e^{\rho t}$; which indicates that $1 + r = e^{\rho}$. The exponential and the geometric rates, which are denoted by ρ and r respectively, differ with $r > \rho$. However, the geometric rate tends to the exponential rate as the length of the unit time period decreases.

Compound Interest and Exponential Growth

To demonstrate the convergence of geometric and exponential growth, and to discover a means of evaluating the natural number e , we can take the analogy of the growth of a fixed-interest financial investment. The aim is to determine the effect of compounding the earnings with the capital at ever-decreasing intervals of time.

If the earnings were compounded with the investment twice a year, then the growth factor would be $(1 + \frac{1}{2}r)^2$. If they were compounded ever quarter, the factor would be $(1 + \frac{1}{4}r)^4$. If the earnings were compounded continuously, then the growth factor would be $\lim(n \rightarrow \infty)(1 + \frac{r}{n})^n$. To evaluate this limit,

we may expand the expression in a power series. Recall that, according to the binomial theorem,

$$(5) \quad (a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \dots + \frac{n(n-1)}{2}a^2b^{n-2} + nab^{n-1} + b^n.$$

This indicates that

$$(6) \quad \left(1 + \frac{r}{n}\right)^n = \left\{1 + n\frac{r}{n} + \frac{n(n-1)}{2}\frac{r^2}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{r^3}{n^3} + \dots\right\}.$$

The consequence is that

$$(7) \quad \lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^n = \left\{1 + r + \frac{r^2}{2!} + \frac{r^3}{3!} + \dots\right\}.$$

Setting $r = 1$ in this expression gives

$$(8) \quad \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = \left\{1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots\right\} = e,$$

and this provides the means of calculating the natural number to any desired degree of accuracy. Finally, by defining $q = n/r$ and by observing that, if r is fixed, then $q \rightarrow \infty$ as $n \rightarrow \infty$, we can see that

$$(9) \quad \lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^n = \left\{ \lim_{q \rightarrow \infty} \left(1 + \frac{1}{q}\right)^q \right\}^r = e^r.$$

Rates of Return

In determining the rate of return of a financial asset, we have the option of calculating either the exponential growth rate ρ or the geometric rate r . Consider the values of the asset at time t and at time $t + 1$. The two values are related as follows:

$$(10) \quad \begin{aligned} y_{t+1} &= y_t(1+r) \\ &= y_t e^\rho = y_t \left(1 + \rho + \frac{\rho^2}{2!} + \frac{\rho^3}{3!} + \dots\right). \end{aligned}$$

The geometric growth rate is calculated as the relative difference of the two values:

$$(11) \quad r = \frac{y_{t+1} - y_t}{y_t}.$$

The exponential rate is calculated as the logarithm of the ratio of the values or, equivalently, as the difference of their logarithms:

$$(12) \quad \rho = \ln \left(\frac{y_{t+1}}{y_t} \right) = \ln(y_{t+1}) - \ln(y_t).$$

When dealing with rates of growth of a few percentage points per period, the difference between the exponential growth rate and the geometric rate is negligible. In effect, the term $\rho^2/2!$ and all succeeding terms in the series expansion of e^ρ given under (13) are of such small values that they can be set to zeros.

Logistic Growth

The rate of increase in a population is the difference between the rate of addition of individuals due to birth B and immigration I and the rate of loss due to death D and emigration E :

$$(13) \quad \frac{dy}{dt} = B + I - D - E.$$

Matters are simplified if we consider a closed population with $I = E = 0$. In most models which are of any interest, the rates of birth and death are functionally related to the size y of the population itself. This is certainly true in the case of exponential growth where the births and deaths are assumed to be directly proportional to y . Thus, with $B = \beta y$ and $D = \delta y$, we have

$$(14) \quad \frac{dy}{dt} = \beta y - \delta y = \rho y,$$

where $\rho = \beta - \delta$. This corresponds to the differential equation of (4) which serves to define the exponential process with the growth rate of ρ . In the closed population, any increase is due entirely to the excess of births over deaths.

It is only in rare cases and for short periods that a population can follow an exponential growth path. Very soon a scarcity of resources or of space, and perhaps an increasing problem of environmental pollution, will slow or arrest the growth. The effect will be achieved either via a decline in the birth rate β or an increase in the death rate δ or in both of these ways. A simple notion is to postulate that the birth rate is a declining linear function of y and that the death rate is an increasing linear function of y :

$$(15) \quad \begin{aligned} \beta &= \lambda_0 + \lambda_1 y, & \lambda_1 &< 0 \\ \delta &= \mu_0 + \mu_1 y, & \mu_1 &> 0. \end{aligned}$$

In that case, there will be a certain population size, say \bar{y} , where the numbers of births and deaths are equal. Thus, if

$$(16) \quad \lambda_0 + \lambda_1 \bar{y} = \mu_0 + \mu_1 \bar{y},$$

then

$$(17) \quad \bar{y} = \frac{\lambda_0 - \mu_0}{\mu_1 - \lambda_1}.$$

Amongst ecologists, this population level is described as the carrying capacity of the environment. It is useful to adopt special symbols both for the

carrying capacity and for the differential between births and deaths at the point where $y = 0$:

$$(18) \quad \gamma = \frac{\lambda_0 - \mu_0}{\mu_1 - \lambda_1}, \quad \rho = \lambda_0 - \mu_0.$$

In these terms, the rate of growth of the population is given by

$$(19) \quad \begin{aligned} \frac{dy}{dt} &= (\beta - \delta)y \\ &= \rho y \left(\frac{\gamma - y}{\gamma} \right). \end{aligned}$$

The expression on the RHS stands for a quadratic function of y whose curve passes through the origin and reaches a maximum at $y = \gamma/2$. This is the so-called logistic model of population growth which is also called the Verhulst–Pearl model.

It is easy to interpret the logistic model which is merely the exponential model of (1) modified by the factor $(\gamma - y)/\gamma$. To understand the effect of this factor, imagine that y is close to zero. Then the value of the factor will be close to unity and the process of growth will be almost exponential. Now imagine that $y = \gamma$, which is to say that the population has reached the carrying capacity of the environment. Then the factor will have the value of zero and there will be no population growth.

The model does not preclude the case where $y > \gamma$. In that case $dy/dt < 0$, and the population will decline toward the level of γ . However, there might be a need to explain how the population has come to exceed the carrying capacity of the environment. The excess population might be due to immigration or it might be due to a reduction that has occurred in the carrying capacity.

The logistic model is uncommon amongst continuous-time models of population dynamics in that its trajectory can be represented by a simple analytic function. Usually such trajectories have to be calculated by a process of numerical integration. The form of the solution for the logistic model depends upon whether the carrying capacity, which represents the steady-state asymptote of the dynamic system, is being approached from above or below. Assuming that $y < \gamma$, a solution is given by the equation

$$(20) \quad y = \frac{\gamma}{1 + e^{-\rho t}} = \frac{\gamma e^{\rho t}}{1 + e^{\rho t}}.$$

Here the second equality comes from multiplying top and bottom by $e^{\rho t}$.

To confirm that the function of (20) does satisfy equation (19), we need to differentiate it:

$$(21) \quad \begin{aligned} \frac{dy}{dt} &= \frac{\gamma \rho e^{-\rho t}}{(1 + e^{-\rho t})^2} \\ &= \rho y \frac{e^{-\rho t}}{1 + e^{-\rho t}}. \end{aligned}$$

Then it only remains to be confirm that

$$(22) \quad \frac{e^{-\rho t}}{1 + e^{-\rho t}} = \frac{\gamma - y}{\gamma},$$

which is easily done.