GEOMETRIC GROWTH AND THE RATE OF INTEREST

Investments

Imagine that a sum of $y_0 = £100$ is invested at an annual rate of interest of $r = 5\%$ per annum. After one year has elapsed, the sum will have grown to $y_0(1+r) = £105$; and the opportunity will arise for deciding how to dispose of the funds. There are two options which might be considered. On the one hand, one might decide leave £100 permanently on account and to treat the interest payment of $r = £5$ as a small annual income or annuity. Alternatively, one might decide leave all of the money on account, and to watch it grow steadily through successive years. In that case, one would be interested in knowing what the investment would amount to after a definite period of years; and, for this purpose, one should need to understand the principle of geometric growth.

Geometric Growth

A process of geometric growth is defined by the equation

$$y_t = y_0(1+r)^t, \tag{1}$$

wherein y_0 stands for the value of the process at time 0 and y_t stands for the value at time t. Here t, which denotes time, takes only integer values.

There are two ways in which the value of y_t can be calculated from those of y_0 and r. The first method uses logarithms:

$$\ln(y_t) = t \times \ln(1+r) + \ln(y_0). \tag{2}$$

The value of y_t is found by applying anti-logarithms to the value obtained from the RHS of the equation. The second method of computing y_t is by iteration. Imagine that $t = 3$. Then

$$y_3 = (1+r)^3 y_0 = \Big\{(1+r)\big[(1+r)\{(1+r)y_0\}\big]\Big\}. \tag{3}$$

This equation can be decomposed into three stages:

$$\begin{aligned} y_1 &= (1+r)y_0, \\ y_2 &= (1+r)y_1, \\ y_3 &= (1+r)y_2. \end{aligned} \tag{4}$$

The generic form of these equations is

$$y_t = (1+r)y_{t-1}; \tag{5}$$

and one can imagine pursuing the iteration though any number of stages. In fact, if we were writing a computer program for the purpose of finding values of y_t, then we should use the iterative scheme on the grounds that it is the quicker one which entails fewer machine operations.

Geometric growth is synonymous with constant proportional growth, which is to say the value of y increases by the same percentage in each time period. To see this, consider

$$\begin{aligned} \frac{\Delta y_t}{y_t} &= \frac{y_{t+1} - y_t}{y_t} \\ &= \frac{y_t(1+r) - y_t}{y_t} = r. \end{aligned} \tag{6}$$

Exponential Growth

The continuous-time analogue of geometric growth is what is known as exponential growth. Such a process is defined by an equation in the form of

$$y_t = y_0 e^{\rho t}. \tag{7}$$

Here y_0 continues to represent the value of the process at time $t = 0$ and y_t stands for the value at time t. The parameter ρ is the exponential growth rate.

In the case of the exponential process, t can assume any real value. However, if t takes an integer value, then equation (7) can be identified with equation (1) of geometric growth by setting

$$e^{\rho} = 1 + r \quad \text{or, equivalently,} \quad \rho = \ln(1 + r). \tag{8}$$

The absolute value of the geometric growth rate exceeds that of the exponential growth rate so that inequality $r > \rho$ holds for positive values. For small values of a few percentage points, the difference between the two rates is negligible. Later, with the help of Taylor's Theorem, we shall establish the precise relationship between r and ρ. For the moment, we need only observe that the geometric rate tends to the exponential rate as the length of the unit time period decreases.

In common with the geometric process, the exponential process may be described as one of constant proportional growth. Thus it is straightforward to confirm that

$$\begin{aligned} \frac{1}{y}\frac{dy}{dt} &= \frac{1}{y_0 e^{\rho t}} \times y_0 e^{\rho t} \rho \\ &= \rho. \end{aligned} \tag{9}$$

Given two successive observation on a process of exponential growth which are separated by a unit time interval, it is straightforward to infer the value of the growth rate. Consider $y_t = \exp\{\rho t\}y_0$ and $y_{t+1} = \exp\{\rho(t+1)\}y_0$. Taking logarithms gives

$$\ln(y_t) = \ln(y_0) + \rho t \quad \text{and} \quad \ln(y_{t+1}) = \ln(y_0) + \rho(t+1). \tag{10}$$

The difference is

$$\rho = \ln(y_{t+1}) - \ln(y_t). \tag{11}$$

The Present Value of an Annuity

An annuity is a sequence of regular payments, made once a year, until the end of the nth year. Usually, such an annuity may be sold to another holder; and, almost invariably, its outstanding value can be redeemed from the institution which has contracted to make the payments. There is clearly a need to determine the present value of the annuity if it is to be sold or redeemed. The principle which is applied for this purpose is that of discounting.

Imagine that a sum of £a is invested for one year at an annual rate of interest of $r \times 100\%$. At the end the year, the principal sum is returned together with the interest via a payment of £$(1 + r)a$. A straightforward conclusion is that £$(1 + r)a$ to be paid one year hence has the value of £a paid today. By the same token, £a to be paid one year hence has a present value of

$$V = \frac{a}{1+r} = a\delta, \qquad \text{where} \quad \delta = \frac{1}{1+r} \quad \text{is the discount rate.} \tag{12}$$

It follows that £a to be paid two years hence has a present value of £$a\delta^2$. More generally, if the sum of £a is to be paid j years hence, then it is worth £$a\delta^j$ today. The present value of an annuity of £a to be paid for the next n years is therefore

$$\begin{aligned} V_n &= a\delta + a\delta^2 + \cdots + a\delta^n \\ &= a\delta\big(1 + \delta + \cdots + \delta^{n-1}\big). \end{aligned} \tag{13}$$

Finding the value V_n is a matter of summing a geometric progression.

The Sum of a Geometric Progression

Consider the indefinite sum $S = \{1 + x + x^2 + \cdots\}$. Given the condition $|x| < 1$, this will have a finite value. Our purpose is to show that, in that case, $S = (1 - x)^{-1}$. The calculation is as follows:

$$\begin{aligned} S &= 1 + x + x^2 + \cdots \\ \underline{xS} &= \underline{\qquad x + x^2 + \cdots} \\ S - xS &= 1. \end{aligned} \tag{14}$$

Then $S(1 - x) = 1$ immediately implies that $S = 1/(1 - x)$.

There is also a formula for the partial sum of the first n terms of the series which is $S_n = 1 + x + \cdots + x^{n-1}$. Consider the following subtraction:

$$\begin{aligned} S &= 1 + x + \cdots + x^{n-1} + x^n + x^{n+1} + \cdots \\ x^n S &= \qquad\qquad\qquad\qquad x^n + x^{n+1} + \cdots \\ \hline S - x^x S &= 1 + x + \cdots + x^{n-1}. \end{aligned}$$

This shows that $S(1 - x^n) = S_n$ whence

$$S_n = \frac{1 - x^n}{1 - x}. \tag{15}$$

The latter result can be applied immediately to the problem of finding the present value of the annuity; for it follows that

$$\begin{aligned} V_n &= a\delta\big(1 + \delta + \cdots + \delta^{n-1}\big) \\ &= a\delta\frac{1 - \delta^n}{1 - \delta}. \end{aligned} \tag{16}$$

A special case is that of a perpetuity which is an annuity to be paid for ever. Its present value is

$$V = \lim_{n\to\infty} a\delta\frac{1 - \delta^n}{1 - \delta} = a\frac{\delta}{1 - \delta} = \frac{a}{r}. \tag{17}$$