RULES OF DIFFERENTIATION

Derivatives

The derivative of a function f at the point x, which is denoted by $f'(x)$, is defined, if it exists, by the the limit

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(x). \tag{1}$$

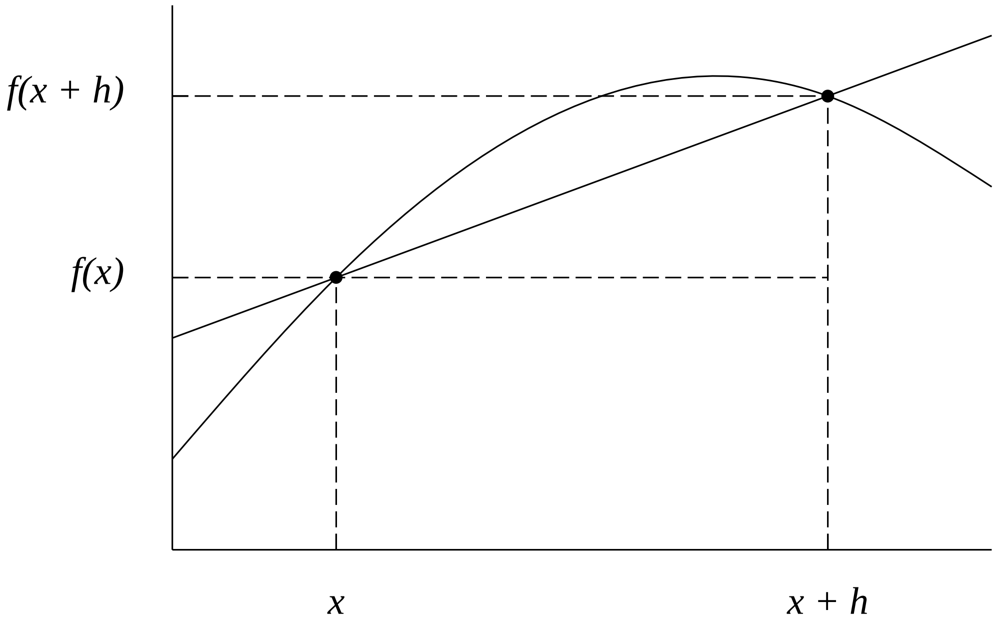

Figure 1.

There are a variety of alternative ways in which the derivative of a function $f(x)$ may be denoted. The delta notation is also commonly employed:

$$\lim_{\delta x\to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = \lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right) = \frac{dy}{dx}. \tag{2}$$

Example. A prototype for differentiation is provided by the function $f(x) = x^2$. In terms of the notation of (2), we have

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\delta x\to 0} \frac{(x+\delta x)^2 - x^2}{\delta x} = \lim_{\delta x\to 0} \frac{(x^2 + 2x\delta x + \{\delta x\}^2) - x^2}{\delta x}\\ &= \lim_{\delta x\to 0} 2x + \delta x = 2x.\end{aligned} \tag{3}$$

Rules of Differentiation

(4) *The Product Rule.* If $u(x)$ and $v(x)$ are functions, continuous in an interval $[a, b]$ with derivatives $u'(a)$ and $v'(a)$ respectively at the point $x = a$, then the derivative of their product $p(x) = u(x)v(x)$ at that point is

$$p'(a) = u(a)v'(a) + v(a)u'(a).$$

The schoolbook method of proving this result is to consider $y = uv$ and to suppose that, when x has a small increment δx, then u has the increment δu and v has the increment δv. Then

$$\begin{aligned} y + \delta y &= (u + \delta u)(v + \delta v) \\ &= uv + u\delta v + v\delta u + \delta u\delta v. \end{aligned} \tag{5}$$

Subtracting $y = uv$ from both sides and dividing by δx gives

$$\frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \frac{\delta u}{\delta x}\delta v. \tag{6}$$

Then it is argued that the ratios of the differentials tend towards the corresponding derivatives as $\delta x \to 0$ and that the final term disappears because $\delta v \to 0$. It should be noted that a proof such as this, which makes no reference to the values of the functions u and v at any specific point, presupposes the existence of the derivatives at all points.

Example. The so-called nominal value $V = pq$ of a manufacturing process is the product of the number of items q produced per unit period, ie. the quantity, and the unit price p. Both price and quantity are liable to change over time, leading to a change in the value of the process. Assuming that $p = p(t)$ and $q = q(t)$ are continuous differentiable functions of time, we have

$$\frac{dV}{dt} = p\frac{dq}{dt} + q\frac{dp}{dt}. \tag{7}$$

The proportional or percentage rate of change of the value is defined by

$$\begin{aligned} \frac{1}{V}\frac{dV}{dt} &= \frac{p}{V}\frac{dq}{dt} + \frac{q}{V}\frac{dp}{dt} \\ &= \frac{1}{q}\frac{dq}{dt} + \frac{1}{p}\frac{dp}{dt}, \end{aligned} \tag{8}$$

which is the sum of the proportional changes in price and in quantity. If prices are increasing at a rate of 10% per annum and the quantity manufactured is growing at a rate of 15% per annum, then the nominal value of the output is changing at a rate of 25% per annum. These are instantaneous rates of change.

Imagine that price were to change over a twelve-month period by 10% and that quantity were to change by 15%. Denote prices at the start of the period by p_0 and at the end of the period by p_1. Use q_0 and q_1 likewise for quantity. Then

$$p_1 = 1.10 \times p_0 \qquad \text{and} \qquad q_1 = 1.15 \times q_0;$$

and the percentage change in the value of output over the period would be

$$\begin{aligned} \frac{V_1 - V_0}{V_0} &= \frac{p_1 q_1 - p_0 q_0}{p_0 q_0} \\ &= 1.10 \times 1.15 - 1.0 = 0.265. \end{aligned}$$

That is to say, there is a $26\frac{1}{2}\%$ increase in value where one might have expected a 25% increase. This appears to contradict our previous finding. The seeming paradox is due to the fact that we are no longer dealing with instantaneous rates of change.

A little algebra may elucidate the matter. Let $\Delta p = p_1 - p_0$, $\Delta q = q_1 - q_0$ and $\Delta V = V_1 - V_0$, Then

$$\begin{aligned} V_1 &= (p_0 + \Delta p)(q_0 + \Delta q) \\ &= p_0 q_0 + p_0 \Delta q + q_0 \Delta p + \Delta p \Delta q. \end{aligned} \tag{9}$$

and, therefore, the proportional change in value is

$$\begin{aligned} \frac{\Delta V}{V_0} &= \frac{p_0 \Delta q + q_0 \Delta p + \Delta p \Delta q}{p_0 q_0} \\ &= \frac{\Delta p}{p_0} + \frac{\Delta q}{q_0} + \frac{\Delta p \Delta q}{p_0 q_0}. \end{aligned} \tag{10}$$

In terms of our example, this equation reads

$$26\tfrac{1}{2}\% = 10\% + 15\% + \big\{10\% \times 15\%\big\}. \tag{11}$$

(12) *The Quotient Rule.* If $u(x)$ and $v(x)$ are functions, continuous in an interval $[a, b]$ with derivatives $u'(a)$ and $v'(a)$, respectively, at the point $x = a$, then the derivative of their quotient $q(x) = u(v)/v(x)$ at that point is

$$q'(a) = \frac{v(a)u'(a) - u(a)v'(a)}{v^2(a)}.$$

One can prove this in terms of differentials. Let $y = u/v$, and consider

$$y + \delta y = \frac{u + \delta u}{v + \delta v}. \tag{13}$$

Subtracting $y = u/v$ from both sides gives

$$\begin{aligned} \delta y &= \frac{u + \delta u}{v + \delta v} - \frac{u}{v} \\ &= \frac{v(u + \delta u) - u(v + \delta v)}{v(v + \delta v)}. \end{aligned} \tag{14}$$

Dividing by δx gives

$$\frac{\delta y}{\delta x} = \frac{1}{v(v + \delta v)} \left\{ v\left(\frac{u + \delta u}{\delta x}\right) + u\left(\frac{v + \delta v}{\delta x}\right) \right\}. \tag{15}$$

Then taking limits gives the derivative:

$$\frac{dy}{dx} = \frac{1}{v^2} \left\{ v\frac{du}{dx} - u\frac{dv}{dx} \right\}. \tag{16}$$

Example. Let Y denote the gross national product (GNP) of a country and let N denote its population. Then $y = Y/N$ is the income per head. The quotient rule indicates that

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{Y}{N}\right) = \frac{1}{N^2} \left\{ N\frac{dY}{dt} - Y\frac{dN}{dt} \right\}. \tag{17}$$

The proportional rate of growth of income per head is

$$\begin{aligned} \frac{1}{y}\frac{dy}{dt} &= \frac{1}{NY} \left\{ N\frac{dy}{dt} - Y\frac{dN}{dt} \right\} \\ &= \frac{1}{Y}\frac{dY}{dt} - \frac{1}{N}\frac{dN}{dt}. \end{aligned} \tag{18}$$

Thus the growth in per capita income is evaluated by subtracting the population growth rate from the growth rate of GNP.