FUNCTIONS, LIMITS AND CONTINUITY

Functions

One of the basic concepts of mathematical analysis, which is also basic to any understanding of economics, is that of a *function.* There are several synonyms for the word function, amongst which the terms *transformation* and *mapping* are the most common. We shall use all three terms interchangeably.

From the point of view of elementary calculus, the notion of a function seems straightforward; and it often remains undefined. A notation such as $y = f(x)$ is usually employed to denote the real-valued product y of a real-valued argument x; and then reliance is placed on our ability to recognise $y = ax + b$, $y = \sin(x)$, $y = x^3$, and so on, as examples of functions to which the notation can be applied

These examples represent rules which associate to a given value of x a unique value of y. The uniqueness of y is crucial. Thus, for example, we cannot associate a unique value of y to the square root of x unless we decide to take either the positive or the negative square root. Also, if the square root is to be real-valued, then the value of x must be non-negative. Thus, the example also suggests that we should take care to define the domain of numbers over which a function such as $y = {}^+\sqrt{x}$ is defined.

Sometimes, a seemingly pedantic distinction is made between the function $f(.)$ and the value $y = f(x)$ which is generated by the function when a specific value of x is supplied to it. Usually, there is no need to make this distinction in elementary applications.

A feature of most elementary functions of real variables is that they have an immediate graphical representation. The values x and y which are associated by a function can be related to the rectangular axes of a Cartesian coordinate system described in a plane. The value of x, which is related to the horizontal axis, is described as the *abscissa,* whilst the value of y, which is related to the vertical axis, is described as the *ordinate.* Together the pair (x, y) constitute the coordinates of a point in the plane.

When we move beyond the realms of elementary calculus, we discover a need for a definition of a function which is at the same time more general and more precise. In particular, it is too restrictive to consider only functions of real variables. Functions of other arguments such as complex variables or vectors are also important. There is no presumption that the product of the function must be the same kind of object as its argument. A good example of a function whose product is in a more extensive set that is its argument arises when we define an ordinary sequence to be a function mapping from the set of integers to the set of real numbers or to a subset of the real numbers.

Eventually we come to recognise that, at its most general, a function must be defined as an association between the objects of two unspecified sets. The defining characteristic of a function is that it associates to every object in the first set $\mathcal{X}$ a unique object in the second set $\mathcal{Y}$. Thus we arrive at the following definition:

(1) A function $f(x)$ is a rule which associates with each element $x \in \mathcal{X}$ a unique value $y = f(x)$. The set $\mathcal{X}$ is described as the *domain* of the function and the set $\mathcal{Y} = \{y = f(x), x \in \mathcal{X}\}$ is described its *range.* The set of all pairs $(x, y) = (x, f(x))$ which are associated by the function are described as the *coordinates* of the function.

Functions of Real Variables

For the time being, we shall confine our attention to ordinary functions of real variables to which the methods of differential calculus can be applied. We shall begin by being precise about some seemingly obvious points, and we shall also define some useful terminology:

(2) *The function $f(x)$ is said be to increasing in the interval* $[a, b]$ if, for every $x_1, x_2 \in [a, b]$ with $x_1 < x_2$, it is found that $f(x_1) \leq f(x_2)$. If it is found that $f(x_1) < f(x_2)$, then $f(x)$ is said to be *strictly increasing* in the interval.

(3) *The function $f(x)$ is said to be monotonic in the interval* $[a, b]$ if it is either strictly increasing or strictly decreasing in the interval.

(4) Let $y = f(x)$ be a function mapping from the set $\mathcal{X} = \{x\}$ to the set $\mathcal{Y} = \{y = f(x); x \in \mathcal{X}\}$ and let $(x, y) = (x, f(x))$ represent the coordinates of the function. If (x_1, y_1) and (x_2, y_2) are coordinates, then the condition $x_1 = x_2$ implies that $y_1 = y_2$. If, conversely, $y_1 = y_2$ implies that $x_1 = x_2$, then there exists an *inverse function* $x = f^{-1}(y)$ which allows the coordinates to be expressed equally as $(x, y) = (f^{-1}(y), y)$.

When a function has an inverse, we can say that there is a *one-to-one* relationship between the elements in the domain of the function and the elements in the range. The existence of the inverse function implies that no two elements in the domain can map into the same element in the range.

The following is a simple but important deduction:

(5) If the domain of $f(x)$ is an interval in which $f(x)$ is monotonic, then $f(x)$ has an inverse function.

Limits

Ordinary differential calculus, which is applied to continuous functions of real variables, depends ultimately on the concept of a limit. The concept of continuity itself depends upon the definition of a limit. The essential definitions often present conceptual difficulties.

It is easiest to understand the concept and the definition of an asymptotic limit; and a good example of the latter is provided by the tendency of the function $y = 1/x$ as x tends to ∞. One should have no difficulty in understanding that, as x increases, the value of y becomes ever closer to zero. In this case, y is not equal to zero for any finite value of x. Nevertheless, we can make y as close to zero as we wish by choosing a value of x which is large enough for the purpose. The following definition describes this kind of circumstance in precise terms:

(6) *The function $f(x)$ is said to tend to a limit ℓ as x tends to infinity,* that is $f(x) \to \ell$ as $x \to \infty$, if, for any number $\epsilon > 0$, a corresponding value q can be found such that, for all $x > q$, we have $\ell - \epsilon < f(x) < \ell + \epsilon$ or we have, equivalently, $|f(x) - \ell| < \epsilon$, where $|z|$ denotes the absolute value of z.

What makes this definition quite easy to understand is the tendency of the function $f(x)$ to approach the asymptote, which is defined by the horizontal axis $y = \ell$, as x increases in an unbounded manner.

A similar definition, the consequences of which may be more difficult to visualise, arises when x approaches a finite value c. Now, instead of watching x travel rapidly outwards into the distance, we have to envisage it travelling with ever-decreasing speed toward a fixed destination. As x approaches c, the value of y approaches its own finite limiting point of ℓ; and, if we are challenged to bring y ever closer to ℓ, then the following definition implies that we can always do so by bringing x closer to c:

(7) *The function $f(x)$ is said to tend to a limit ℓ as x tends to c,* that is $f(x) \to \ell$ as $x \to c$ if, for any number $\epsilon > 0$, a number $\eta > 0$ can be found such that $|f(x) - \ell| < \epsilon$ whenever $|x - c| < \eta$.

The foregoing definition makes no mention of the direction in which the value c is approached by x. The implication is that the statement holds true regardless of the direction. For some purposes, it is important to be specific about the direction. Thus

(8) *The function $f(x)$ is said to tend to ℓ as x tends to c from below,* that is $f(x) \to \ell$ as $x \to c-$ or, in an alternative notation, $x \uparrow c$, if, for any number of $\epsilon > 0$, a value η can be found such that $|f(x) - \ell| < \epsilon$ whenever $x \in (c - \eta, c)$.

A similar definition can be given for a limit of $f(x)$ as x approaches c from above. The two limits of the function $f(x)$ at the point c may be denoted by $\lim(x \to c+)f(x)$ and $\lim(x \to c-)f(x)$, and they need not be equal. If the limits are unequal, then the point c is called a jump-point and the function has a jump or a *saltus* at that point.

These distinctions are helpful when it comes to providing a definition of continuity.

Continuity

A continuous function is one whose graph contains no sudden jumps. Roughly speaking, it is one which can be drawn without lifting the pen from the paper. A more precise definition is as follows:

(9) *The function $f(x)$ is said to be continuous at the point $x = c$* if $\lim(x \to c+)f(x)$ and $\lim(x \to c-)f(x)$ exist and are equal to $f(c)$.

Derivatives

(10) The derivative of a function $f(x)$ at the point a, which is denoted by $f'(a)$, is defined, if it exists, by the the limit

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = f'(a).$$

(11) When a function $f(x)$ is continuous and differentiable in the interval (a, b), its derivative at a point in $x \in (a, b)$ is commonly denoted by

$$\lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} = \lim_{\delta x \to 0} \left(\frac{\delta y}{\delta x} \right) = \frac{dy}{dx}.$$

(12) *The Product Rule.* If $u(x)$ and $v(x)$ are functions, continuous in an interval $[a, b]$ with derivatives $u'(x)$ and $v'(x)$ respectively, then the derivative of their product $p(x) = u(x)v(x)$ is

$$p'(x) = u(x)v'(x) + v(x)u'(x).$$

(13) *The Quotient Rule.* If $u(x)$ and $v(x)$ are functions, continuous in an interval $[a, b]$ with derivatives $u'(x)$ and $v'(x)$ respectively, then the derivative of their quotient $q(x) = u(v)v(x)$ is

$$q'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{v^2(x)}.$$

(14) *The Chain Rule.* If $y = f(x)$ and $z = g(y)$ are continuous differentiable functions, whose derivatives are $f'(x) = dy/dx$ and $g'(y) = dz/dy$ respectively, then the derivative of their composition $z(x) = g\{f(x)\}$ is

$$\frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx}.$$