NUMBERS, LIMITS AND CONTINUITY

Numbers

The concept of a real number, which is entailed in many of the daily activities of science and commerce, if far more elusive that one in inclined to imagine if one takes a common-sense approach to ordinary matters of calculation. In fact, the concept of a real number seems, on closer examination, to be so elusive and unreal that the 19th-century German mathematician Dedekind declared that, if the natural numbers were created by God, then, surely, man must have invented the other numbers—which is to say that the real numbers must be some figment of the human imagination.

Integer Numbers. By talking of the natural numbers, one is usually referring to the set of positive integers $\mathcal{Z} = \{1, 2, 3, \ldots\}$. Some people would use the phrase to denote the set of all integers $\mathcal{I} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$. This is surely a misnomer. The number zero 0 is a sophisticated invention of Renaissance commercial arithmetic. The unnatural concept of a negative number also originated in the same era when it was first used in denoting financial deficits. The advantage of negative numbers in axiomatising the rules of arithmetic is that it enables us to dispense with the redundant operation of subtraction by replacing all subtractions by additions of negative numbers.

Rational Numbers. The natural numbers are inadequate for the purposes of measurement where fractions of units are required. The requirements of measurement lead to the invention of rational numbers. A rational number $x = p/q$ is defined as the ratio two integers p and q, at least one of which is odd and one of which one may be negative. It is understood that p and q have no factors in common. Of course, if both p and q are even, then they can both be reduced by a factor of 2. Also, we might insist that, if one of the numbers is negative, then it must be p, since $p/(-q) = -p/q$. (we have already disallowed $(-p)/(-q)$ since there is no reason to write this in place of p/q, although most computer languages will allow it so long as the denominator is in parentheses).

The rational numbers might seem to be adequate for the purposes of measurement since they seem, at first sight, to correspond to all of the points on a straight line. In particular, they cover the real line densely. By this it is meant that, between any two points on the real line corresponding to a pair of rational numbers, be they ever so close, there can be found an infinity of rational numbers.

There is no need for a sophisticated demonstration of this point. Instead we can use a simple construction. Consider the rational numbers a and b with $p_1/q_1 = a < b = p_2/q_2$. The object is to find another number inside the interval bounded by a and b. Observe that $p_1q_2 < p_2q_1$. Moreover, if $q_2.abc$ is formed by appending a string of decimal digits to q_2, then also $p_1(q_2.abc) < p_2q_1$ and, clearly,

$$\frac{p_1}{q_1} < \frac{p_1(q_2.abc)}{q_1 q_2} < \frac{p_2}{q_2}. \tag{21}$$

The term in the middle of the inequalities is a rational number, as is seen when the numerator and denominator are raised by a factor of 1,000 and any resulting common factors are eliminated.

In practice, the rational numbers, in the form of terminated decimals, are the only numbers which are available for the purposes of calculation and measurement. However, it is easy to see why they are inadequate in theory. Consider a simple problem in building construction which is to form an isosceles right-angle triangle from thee pieces of timber for use as a roof truss of as a buttress of some sort. If the two timbers at right angles are of unit length, then the length of the other timber, which forms the hypotenuse or the tie-beam, is of length $\sqrt{2}$; and this is not a rational number.

Proof. If the square root of 2 were a a rational number, then it should be possible to find integer numbers p and q, with no factor in common, satisfying the equation $(p/q)^2 = 2$, which is equivalent to the equation $p^2 = 2q^2$. Now, if it is an integer, p must be a factor of the expression of the RHS. That is to say, it is either a factor of 2 or of q^2. But p is not a factor of 2 since the only integers which divide 2 are 1 and 2, and p is neither of these. Also, p is not a factor of q^2 since p and q have no common factors. It follows that p is not an integer, and hence $(p/q) = \sqrt{2}$ is not a rational number.

The Enumeration of Rational Numbers. In spite of their density over the line, the rational numbers are capable of enumeration. That is to say, by virtue of a simple construction, they can be placed in a one-to-one correspondence with the set of natural numbers $\mathcal{Z} = \{1, 2, 3, \ldots\}$. Consider the following table:

(22)

	1	2	3	4	5
1	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{5}{1}$
2	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$	$\frac{5}{2}$
3	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\frac{5}{3}$
4	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$
5	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{5}{5}$

Along the top of the table are the numerators of the rational numbers. In the left margin of the table are the denominators. The table can be enlarged indefinitely by extending its width and its depth. By reading along successive diagonals, the following sequence can be constructed:

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{2}{2}, \frac{3}{1}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1}, \ldots \tag{23}$$

From this sequence, the redundant elements such as $\frac{2}{2}$, $\frac{3}{3}$ etc, can be struck out. By this process, which continues indefinitely, the positive rational numbers can be generated. The resulting sequence bears a one-to-one correspondence with the set of positive integers.

Irrational Numbers. Ancient Greek mathematicians, such as Pythagoras, were inclined to treat the irrational numbers such as $\sqrt{2}$, π and e as handful of oddities and, at times, they even asserted that these are not numbers at all. However, there are strong arguments for including the irrationals in any system of numbers. Consider a number such as $\sqrt{2}$. This can be placed inside an interval bounded by two rational numbers, say 1 and 4. (These are positive integers, but we can also describe them as rational since the set of rational numbers includes all of the integers). We can use the first of these numbers as the starting point of an ordinary arithmetic process for the extraction of a square root. The process generates the following sequence of rational numbers:

$$\mathcal{A}: \quad 1,\ 1.4,\ 1.41,\ 1.414,\ 1.4142, \ldots \tag{24}$$

of which the squares are

$$\mathcal{B}: \quad 1,\ 1.96,\ 1.9881,\ 1.999396,\ 1.99996164, \ldots \tag{25}$$

These are all less that 2; but they are becoming ever closer to it. Adding a unit to the final decimal figure of each element of the sequence $\mathcal{A}$ generates the sequence

$$\mathcal{C}: \quad 2,\ 1.5,\ 1.42,\ 1.415,\ 1.4143, \ldots \tag{26}$$

of which the squares are

$$\mathcal{D}: \quad 4,\ 2.25,\ 2.0164,\ 2.002225,\ 2.00024449, \ldots \tag{27}$$

These are all greater that 2 but are becoming ever closer to it. They are also becoming ever closer to elements of the sequence $\mathcal{B}$. In fact, by extending these processes, we can make elements of $\mathcal{A}$ and $\mathcal{C}$ and those of $\mathcal{B}$ and $\mathcal{D}$ as close to each other as we wish, but not equal. We have no problem in declaring the 2 is the common limit of $\mathcal{B}$ and $\mathcal{D}$. It therefore seems almost unavoidable that we should postulate the existence of a number $\sqrt{2}$ which is the common limit of $\mathcal{A}$ and $\mathcal{C}$.

The foregoing is an argument in favour of including the irrationals in our number system which is based only on simple arithmetic. When we consider the requirements of simple algebra, there is no avoiding the irrationals. It is an easily proven fact that there is no rational number whose square is m/n, where m and n are integers, unless m and n are also perfect squares. Therefore, without the irrational numbers, the business of solving quadratic equations, for example, could hardly begin.

The final point to be made regarding the irrational numbers concerns their density over the real line. Far from representing a mere handful of pathological cases, as the ancient Greeks believed, the irrational numbers are far more prevalent that the rationals. In fact, they cover the real line so densely that there is no possibility of enumerating them. As we have seen, There are infinitely many rational numbers in a given interval on the line. However, this is a so-called *denumerable infinity.* The infinity which characterises the set of irrational numbers in the same interval is of a far higher order. It is called a *non-denumerable infinity.*

The rational and irrational numbers, taken together, constitute the set of real numbers. It is in reference to the set of real number that the majority of the concepts of calculus and analysis are developed, including those of limits and of continuity.

Intervals and Inequalities. Before elaborating some simple concepts in analysis, it helps to refine the concept of an interval of the real line and to develop some associated notations. For this purpose, we need to employ the notation of inequalities. If a and b are two numbers such that a is strictly less than b, then we may denote this circumstance by writing $a < b$ or, equivalently, $b > a$ (b is strictly greater that a). These are a case of a *strong inequality.* If, on the other hand, it is certain only that a does not exceed b, then we write $a \leq b$ (a is less than or equal to b) or, equivalently, $b \geq a$ (b is greater than or equal to a). These are cases of a *weak inequality.*

(28) If a and b are two numbers such that $a < b$, then the set of numbers x satisfying $a \leq x \leq b$ is described as a closed interval from a to b, denoted by $[a, b]$.

A closed interval is one which includes its endpoints, and the brackets $[,]$ signify its closure.

(29) If a and b are two numbers such that $a < b$, then the set of numbers x satisfying $a < x < b$ is described as an open interval from a to b denoted by (a, b).

An open interval is one which does not include its endpoints, and this is signified by the use of parentheses $(,)$ instead of brackets.

It is also common to define intervals which are half-open and half-closed, Thus

(30) $$(a, b] = \{x;\ a < x \leq b\} \quad \text{and} \quad [a, b) = \{x;\ a \leq x < b\}.$$

Examples of such intervals arise in integration. We commonly define the integral of the function $f(x)$ from a to b by the expression

$$\int_a^b f(x)dx = \int_{-\infty}^b f(x)dx - \int_{-\infty}^a f(x)dx. \tag{31}$$

This integral, which is defined over the interval $(a, b]$, is obtained by subtracting the integral over $(-\infty, a]$ from the integral over $(-\infty, b]$. If we want to reconstruct an integral over an given interval from the integral over its constituent sub-intervals, then we ought to be punctilious in the matter of assigning the end-points correctly to the intervals. Admittedly, if $f(x)$ is a continuous function, then no significant penalty is liable to result from getting things slightly wrong.

In other circumstances the correct assignment of the end points is crucial. For example, in an old system of degree assessment, examination scripts were marked out of 100 and were then graded as follows:

$$\begin{aligned} &\mathrm{A} \in [75, 100], \\ &\mathrm{B} \in [65, 75), \\ &\mathrm{C} \in [55, 65), \\ &\mathrm{D} \in [45, 55), \\ &\mathrm{E} \in [35, 45), \\ &\mathrm{F} \in [0, 35). \end{aligned} \tag{32}$$

Then, when there were five papers of a given grade, the grade was automatically translated into a honours class: A corresponding to first-class honours, B corresponding to the upper-second class honours, and so on. Lengthy discussions could arise in the examiners' meeting on the issue of whether a given script merited a mark of 65 or a mark below 65, albeit within an iota (or an epsilon ϵ) or it. Such fine points were all-important in determining the class of a degree.