TAYLOR'S THEOREM AND SERIES EXPANSIONS

Taylor's Theorem. If f is a function continuous and n times differentiable in an interval $[x, x+h]$, then there exists some point in this interval, denoted by $x + \lambda h$ for some $\lambda \in [0,1]$, such that

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \cdots$$
$$\cdots + \frac{h^{(n-1)}}{(n-1)!}f^{(n-1)}(x) + \frac{h^n}{n!}f^n(x+\lambda h).$$

If f is a so-called analytic function of which the derivatives of all orders exist, then one may consider increasing the value of n indefinitely. Thus, if the condition holds that

$$\lim_{n\to\infty} \frac{h^n}{n!}f^n(x) = 0,$$

which is to say that the terms of the series converge to zero as their order increases, then an infinite-order Taylor-series expansion is available in the form of

$$f(x+h) = \sum_{j=0}^{\infty} \frac{h^j}{j!}f^j(x).$$

This is obtained simply by extending indefinitely the expression from Taylor's Theorem. In interpreting the summary notation for the expansion, one must be aware of the convention that $0! = 1$.

A Taylor-series expansion is available for functions which are analytic within a restricted domain. An example of such a function is $(1-x)^{-1}$. The function and its derivatives are undefined at the point $x = 1$. Nevertheless, Taylor-series expansions exists for the function at all other points and for all $|h| < 1$. Another example is provided by the function $\log(x)$ which is defined only for strictly positive numbers $x > 0$.

The expression for Taylor's series given above may be described as the expansion of $f(x+h)$ about the point x. It is also common to expand a function $f(x)$ about the point $x = 0$. The resulting series is described as Maclaurin's series:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \cdots.$$

We shall give a number of examples of such expansions; all of which may be memorised profitably.

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Example 1. $\log(1+x)$

$$\begin{aligned} f(x) &= \log(1+x) & f(0) &= 0 \\ f'(x) &= \frac{1}{1+x} & f'(0) &= 1 \\ f''(x) &= \frac{-1}{(1+x)^2} & f''(0) &= -1 \\ f'''(x) &= \frac{2}{(1+x)^3} & f'''(0) &= 2 \\ f^{(4)}(x) &= \frac{-2.3}{(1+x)^4} & f^{(4)}(0) &= -6 \end{aligned}$$

It follows that, for $|x| < 1$—which is the necessary and sufficient condition for the convergence of the series—we have

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$

Example 2. e^x

$$\begin{aligned} f(x) &= e^x & f(0) &= 1 \\ f^{(n)}(x) &= e^x & f^{(n)}(0) &= 1 \end{aligned}$$

It follows that, for $|x| < 1$, we have

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$$

Example 3. $\sin x$

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \\ f^{(5)}(x) &= \cos x & f^{(5)}(0) &= 1 \end{aligned}$$

It follows that, for $|x| < 1$, we have

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$$

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Example 4. $\cos x$

$$\begin{aligned} f(x) &= \cos x & f(0) &= 1 \\ f'(x) &= -\sin x & f'(0) &= 0 \\ f''(x) &= -\cos x & f''(0) &= -1 \\ f'''(x) &= \sin x & f'''(0) &= 0 \\ f^{(4)}(x) &= \cos x & f^{(4)}(0) &= 1 \end{aligned}$$

It follows that, for $|x| < 1$, we have

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots .$$

Example 5. $1/(1-x)$

$$\begin{aligned} f(x) &= (1-x)^{-1} & f(0) &= 1 \\ f^{(n)}(x) &= n!(1-x)^{-n} & f^{(n)}(0) &= n! \end{aligned}$$

It follows that, for $|x| < 1$, we have

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots .$$

The last of these examples is a familiar one which may have been encountered for the first time in the context of the summation of a geometric progression. There is a wide variety of alternative ways of demonstrating this expansion. Amongst these is simple long division:

$$\begin{array}{r l} & \underline{\;1 + x + x^2 + \cdots\;} \\ 1 - x \,\big) & 1 \\ & \underline{1 - x} \\ & \quad x \\ & \quad \underline{x - x^2} \\ & \qquad x^2 \\ & \qquad x^2 - x^3 \end{array}$$

We can also proceed in the opposite direction. That is to say, we can evaluate $S = \{1 + x + x^2 + \cdots\}$ to show that $S = (1-x)^{-1}$. The calculation is as follows:

$$\begin{aligned} S &= 1 + x + x^2 + \cdots \\ xS &= \quad\;\; x + x^2 + \cdots \\ \hline S - xS &= 1. \end{aligned}$$

Then $S(1-x) = 1$ immediately implies that $S = 1/(1-x)$.

There is also a formula for the partial sum of the first n terms of the series, which is $S_n = 1 + x + \cdots + x^{n-1}$. Consider the following subtraction:

$$\begin{array}{l} \quad S = 1 + x + \cdots + x^{n-1} + x^n + x^{n+1} + \cdots \\ x^n S = \qquad\qquad\qquad\quad\;\; x^n + x^{n+1} + \cdots \\ \hline S - x^n S = 1 + x + \cdots + x^{n-1}. \end{array}$$

This shows that $S(1-x^n) = S_n$, whence

$$S_n = \frac{1-x^n}{1-x}.$$

Example. An annuity is a sequence of regular payments, made once a year, until the end of the nth year. Usually, such an annuity may be sold to another holder; and, almost invariably, its outstanding value can be redeemed from the institution which has contracted to make the payments. There is clearly a need to determine the present value of the annuity if it is to be sold or redeemed. The principle which is applied for this purpose is that of discounting.

Imagine that a sum of £A is invested for one year at an annual rate of interest of $r \times 100\%$. At the end the year, the principal sum is returned together with the interest via a payment of £$(1+r)A$. A straightforward conclusion is that £$(1+r)A$ to be paid one year hence has the value of £A paid today. By the same token, £A to be paid one year hence has a present value of

$$V = \frac{A}{1+r} = A\delta, \qquad \text{where} \quad \delta = \frac{1}{1+r} \quad \text{is the discount rate.}$$

It follows that £A to be paid two years hence has a present value of £$A\delta^2$. More generally, if the sum of £A is to be paid n years hence, then it is worth £$A\delta^n$ today.

The present value of an annuity of £rA to be paid for the next n years is therefore

$$\begin{aligned} V_n &= Ar(\delta + \delta^2 + \cdots + \delta^n) = Ar\delta\bigl(1 + \delta + \cdots + \delta^{n-1}\bigr) \\ &= Ar\delta\frac{1-\delta^n}{1-\delta} = A(1-\delta^n), \qquad \text{since} \quad \frac{\delta}{1-\delta} = r. \end{aligned}$$

If the principal sum is to be repaid at the end of the nth year, then the present value of the contract will be

$$A(1-\delta^n) + A\delta^n = A,$$

which is precisely equal to the value of the sum that is to be invested.

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