CONTINUOUS-TIME STOCHASTIC PROCESSES

Discrete-Time Random Walk The concept of a Wiener process is an extrapolation of that of a discrete-time random walk. A standardised random walk is a process that is defined over the set of integers $\{t = 0, \pm 1, \pm 2, \ldots\}$, which represent dates separated by a unit time interval. It may be denoted by $w(t) = \{w_t; t = 0, \pm 1, \pm 2, \ldots\}$, and it can be represented by the equation

$$w_{t+1} = w_t + \varepsilon_{t+1}, \tag{1}$$

wherein ε_t is an element of a sequence $\varepsilon(t) = \{\varepsilon_t; t = 0, \pm 1, \pm 2, \ldots\}$ of independently distributed standard normal random elements with a mean value of of $E(\varepsilon_t) = 0$ and a variance of $V(\varepsilon_t) = 1$, for all t, which is described as a white-noise process.

By a process of back-substitution, the following expression can be derived:

$$w_t = w_0 + \big\{\varepsilon_t + \varepsilon_{t-1} + \cdots + \varepsilon_1\big\}. \tag{2}$$

This depicts w_t as the sum of an initial value w_0 and of an accumulation of stochastic increments. If w_0 has a fixed finite value, then the mean and the variance of w_t, conditional of this value, are be given by

$$E(w_t|w_0) = w_0 \quad \text{and} \quad V(w_t|w_0) = t. \tag{3}$$

There is no central tendency in the random-walk process; and, if its starting point is in the indefinite past rather than at time $t = 0$, then the mean and variance are undefined.

To reduce the random walk to a stationary stochastic process, it is necessary only to take its first differences. Thus

$$w_{t+1} - w_t = \varepsilon_{t+1}. \tag{4}$$

We should also observe that we can take larger steps through time without fundamentally altering the nature of the process. Let h be any integral number of time periods. Then

$$w_{t+h} - w_t = \sum_{j=1}^{h} \varepsilon_{t+j} = \zeta_{t+h}\sqrt{h}, \tag{5}$$

where ζ_{t+h} is an element of a sequence $\{\zeta_{t+jh}; j = 0, \pm 1, \pm 2, \ldots\}$ of independently and identically distributed standard normal variates. The factor $\sqrt{h}$ has

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entered the equation for the reason that we are now taking steps through time of length h whereas, previously, we were taking steps of unit length.

A random walk, as the name implies, have a tendency to wander haphazardly. However, if the variance of the white-noise process that is driving the random walk is small, then the values of the stochastic increments will also be small and the random walk will wander slowly.

A first-order random walk over a surface is know as Brownian motion. For a physical example of Brownian motion, one can imagine small particles, such a pollen grains, floating on the surface of a viscous liquid. The viscosity might be expected to bring the particles to a halt quickly if they are in motion. However, if the particles are very light, then they will dart hither and thither on the surface of the liquid under the impact of its molecules, which are themselves in constant motion. The term *Brownian motion* has been adopted to describe a univariate processes.

Wiener Processes A Wiener process is the consequence of allowing the intervals of a discrete-time random walk to tend to zero. The dates at which the process is defined become a continuum. The result is a process that is continuous almost everywhere but nowhere differentiable.

The Wiener process has all of the characteristics of the random walk process that has been described above. When sampled at regular intervals, it has the same mathematical description as a discrete-time process. However, whereas the random walk is defined only on the set of integers, the Wiener process is defined for all points on a real line that represents continuous time.

The generalisation can be achieved by replacing the integer h of equation (5) by an increment dt that can take infinitesimally small values. The equation can be rewritten accordingly as

$$dw(t) = w(t + dt) - w(t) = \zeta(t + dt)\sqrt{dt} \tag{6}$$

This equation describes a standard Wiener process. The process fulfils the following conditions:

(a) $w(0) = 0$,

(b) $E\{w(t)\} = 0$, for all t,

(c) $w(t)$ is normally distributed,

(d) $dw(s), dw(t)$ for all $t \neq s$ are independent stationary increments,

(e) $V\{w(t + h) - w(t)\} = h$ for $h > 0$.

The assumption (a) that the initial value $w(0)$ is zero is unrestrictive, so long as it can be assumed that the process takes a finite value at time $t = 0$. For that value can subtracted from the process to yield condition (a).

Arithmetic Brownian Motion The standard Wiener process is inappropriate to much of financial modelling. However, some quite general continuous stochastic processes can be derived that are functions of a standard Wiener process.

A straightforward generalisation corresponds to a so-called random walk with drift. In discrete time, this can be represented by the equation

$$x(t+1) = x(t) + \mu + \sigma\varepsilon(t+1). \tag{7}$$

The continuous-time analogue of this process is described by

$$dx(t) = \mu dt + \sigma dw(t). \tag{8}$$

A generalisation of the latter is the Ito process, where the drift parameter μ and the variance or volatility parameter σ^2 become time-dependent functions of the level of the process:

$$dx(t) = \mu(x,t)dt + \sigma(x,t)dw(t). \tag{9}$$

Geometric Brownian Motion The domain of a normally distributed random variable is the entire real line, which extends from $-\infty$ to $+\infty$. Many variables, such as those that represent physical quantities, are constrained to lie in the interval $[0,\infty)$. Therefore, it is inappropriate to describe them with a model that corresponds to a linear function of a normal random variable. In finance, there is an evident constraint that nominal interest rates must be nonnegative. Also, asset values cannot become negative.

Such difficulties in modelling can sometimes be overcome by replacing the variables in question by their logarithms. The logarithmic transformation maps from the interval $[0,\infty)$ to the interval $(-\infty,\infty)$.

The logarithmic version of the random walk is described by the following equation:

$$\ln x(t+1) = \ln x(t) + \sigma\varepsilon(t+1). \tag{10}$$

The corresponding continuous-time version can be written as

$$d\{\ln x(t)\} = \sigma dw(t). \tag{11}$$

Given that

$$\frac{d}{dt}\ln x = \frac{1}{x}\frac{dx}{dt} \qquad \text{or} \qquad d\{\ln x(t)\} = \frac{dx}{x},$$

it follows that equation (11) can also be written as

$$dx = \sigma x dw(t). \tag{12}$$

This equation might be used in describing the trajectories of financial assets. Observe that the process has an absorbing barrier at zero. That is to say, if $x = 0$ at any time, then it will remain at that value thereafter.

A equation that can be applied more generally in describing financial assets is one that incorporates a drift term:

$$dx = x\mu dt + \sigma x dw(t). \tag{13}$$

On the strength of the preceding reasoning, it might be imagined that this is synonymous with the process described by the equation $d\{\ln y(t)\} = \mu dt + \sigma dw(t)$. It is interesting to discover that this is not the case.

To find the logarithm of the process described by (13), we must use Ito's Lemma. This indicates that, for any process described by equation (9), and for any continuous differentiable function $f(x)$, there is

$$df(x,t) = \left\{\mu(x,t)\frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} + \sigma^2(x,t)\frac{1}{2}\frac{\partial^2 f}{\partial x^2}\right\}dt + \sigma(x,t)\frac{\partial f}{\partial x}dw. \tag{14}$$

We take $f(x) = \ln x$. Also, equation (13) is assimilated to equation (9) by setting $\mu(x,t) = x\mu$ and $\sigma(x,t) = \sigma x$. Then,

$$\frac{\partial f}{\partial x} = \frac{1}{x}, \qquad \frac{\partial f}{\partial t} = 0 \qquad \text{and} \qquad \frac{1}{2}\frac{\partial^2 f}{\partial x^2} = -\frac{1}{2x^2}. \tag{15}$$

Therefore, from Ito's lemma, we obtain

$$\begin{aligned} d\ln x &= \left\{\frac{1}{x}x\mu - \frac{1}{2x^2}\sigma^2 x^2\right\}dt + \frac{1}{x}\sigma x dw \\ &= \left\{\mu - \frac{\sigma^2}{2}\right\}dt + \sigma dw. \end{aligned} \tag{16}$$

Here, the drift parameter is $\mu - \sigma^2/2$, where we might have expected to find just μ. However, on reflection, it seems reasonable that the drift parameter, which denotes a rate of exponential growth, should be diminished by an increase in the volatility.

The cumulation of $\ln x$ over a finite time inteval $[t, t+h]$ generates a normally distributed random variable

$$(\ln x_{t+h} - \ln x_t) = \ln\left(\frac{x_{t+h}}{x_t}\right) \sim N\left(\left\{\mu - \frac{\sigma^2}{2}\right\}h, \sigma^2 h\right). \tag{17}$$

If $\ln y \sim N(\mu, \sigma^2)$ is a normally distributed random variable, then y is said to have a log-normal distribution. For any such log-normal variable y, it is the case that

$$\begin{aligned} \ln\{E(y)\} &= E\{\ln y\} + \frac{1}{2}V\{\ln y\} \\ &= \mu + \frac{\sigma^2}{2}. \end{aligned} \tag{18}$$

The equality must be rearranged to give the expected value in (17). Observe that, since the logarithmic transformation in nonlinear, it follows that the logarithm of the expectation is not the expectation of the logarithm.

We shall make the assumption that the spot price S of a financial asset follows a geometric Brownian walk. This leads us to write

$$d \ln S = \left\{ \mu - \frac{\sigma^2}{2} \right\} dt + \sigma dw. \tag{19}$$

Over a finite interval from $t = 0$ to $t = \tau$, this gives rise to a normally distributed random variable

$$(\ln S_\tau - \ln S_0) = \ln \left(\frac{S_\tau}{S_0} \right) \sim N\left(\left\{ \mu - \frac{\sigma^2}{2} \right\} \tau, \sigma^2 \tau \right). \tag{20}$$

Also, consider the formula

$$S_\tau = S_0 e^{\rho\tau}, \tag{21}$$

from which

$$\rho = \frac{1}{\tau} \ln \left(\frac{S_\tau}{S_0} \right). \tag{20}$$

It follows directly from (20) that

$$\rho \sim N\left(\left\{ \mu - \frac{\sigma^2}{2} \right\}, \frac{\sigma^2}{\tau} \right). \tag{22}$$

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