PRESENT VALUES

The Initial Value of a Forward Contract. One of the parties to a forward contract assumes a *long position* and agrees to buy the underlying asset at a certain price on a certain specified future date denoted $t = \tau$. The other party assumes a short position and agrees to sell the asset on the same date. The date when the contract is made is $t = 0$. The agreed settlement price is $K_\tau = F_{\tau|0}$, where $F_{\tau|0}$ denotes the price at time $t = 0$ for a delivery of the asset at time $t = \tau$.

Let $t = 0$ be the current time so that S_0 is the spot price of an asset. Let the current risk-free rate of compound interest be r. Then, the spot price and the forward price are related by the formulae

$$\text{(i)} \quad F_{\tau|0} = S_0 e^{r\tau} \qquad \text{and} \qquad \text{(ii)} \quad S_0 = F_{\tau|0} e^{-r\tau}.$$

Here, we understand that the forward price $F_{\tau|0}$ must be discounted by the factor $e^{-r\tau}$ to equate it to the present value of S_0. Equally, if the sum of S_0 were to be invested for τ periods under a regime of compound interest, then it would grow to $S_0 e^{r\tau}$. To establish the necessity of the relationships, we may consider how, in their absence, there would be possibilities for arbitrage, which may be ruled out by assumption.

Imagine that $S_0 e^{r\tau} > F_{\tau|0}$. An investor could sell the asset today for S_0 and invest the proceeds to derive a sum of $S_0 e^{r\tau}$ at time τ. At the same time, he could enter a long forward contract to buy the asset at time τ for $F_{\tau|0}$. In this way, he would derive a riskless arbitrage profit of $S_0 e^{r\tau} - F_{\tau|0} > 0$. This should not be possible under the assumption that all arbitrage opportunities are immediately exploited and that they vanish on the instant.

Imagine, conversely, that $F_{\tau|0} > S_0 e^{r\tau}$. The investor could borrow S_0 today and undertake to repay $S_0 e^{r\tau}$ at time τ. Then, he could buy the asset for S_0 and enter a short forwards contract to sell it for $F_{\tau|0}$ at time τ. In this way, he could derive a riskless arbitrage profit of $F_{\tau|0} - S_0 e^{r\tau} > 0$. Such a possibility is also ruled out. If neither $S_0 e^{r\tau} > F_{\tau|0}$ nor $F_{\tau|0} > S_0 e^{r\tau}$ are possible, then it must be true that $F_{\tau|0} = S_0 e^{r\tau}$.

The Current Value of a Forward Contract. After a forward contact has been established there may be a marked divergence between the current forward price and the settlement price. Image that the contract was made at time $t = 0$, when the condition $F_{\tau|0} = S_0 e^{r\tau}$ would have prevailed, which links the spot price S_0 to the forwards price $F_{\tau|0}$ via the risk free rate of interest r. At a subsequent date t, the value of a long forward contract to accept delivery of the asset at time τ will be

$$f_t = (F_{\tau|t} - K_\tau) e^{-r(\tau - t)},$$

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where $\tau - t$ is the time remaining until the settlement date.

The contract matures at time $t = \tau$. At that time, there will be $F_{\tau|\tau} = S_\tau$, which is to say that the forward price will coincide with the spot price. Both K_τ and S_τ will be present values. Therefore the profit or loss for the long position will be

$$f_\tau = S_\tau - K_\tau.$$

Forward Price of an Income Bearing Asset. Consider an asset that generates an income stream. Let $t = 0$ be the current time, and let $t = \tau$ be the time at which the asset is to be sold under a forwards contract. The present value of the income stream from now till then can be denoted by $I_{\tau|0}$. Then, the forward price of the asset is

$$F_{\tau|0} = (S_0 - I_{\tau|0})e^{r\tau}.$$

This follows since the income stream up to time τ, which will not be received by the party purchasing the forward contract, must be discounted to its present value and subtracted from the spot price. The factor $e^{r\tau}$ serves to carry $S_0 - I_{\tau|0}$ forward from time $t = 0$ to time $t = \tau$.

To confirm this formula, consider buying a unit of the asset at time $t = 0$ for S_0. This is certain to lead to a payment of $F_{\tau|0}$ when delivered to another party in fulfilment of the forward contract. The present value of this payment is $F_{\tau|0}e^{-r\tau}$, which can be added to the present value $I_{\tau|0}$ of the income stream up to time τ. This sum must equal the spot price. So $S_0 = I_{\tau|0} + F_{\tau|0}e^{-r\tau}$, which is equivalent to the stated result.

Forward Price of an Asset Yielding Dividends. A dividend is calculated as a fixed percentage of the asset price. Let the dividend be paid continuously at an annual rate of q and imagine that the payments are compounded with the asset. Let the risk free rate of interest be r. Then, the forwards price of an investment asset providing a continuous dividend yield at the rate q is

$$F_{\tau|0} = S_0 e^{(r-q)\tau}.$$

To rationalise this, consider buying N units of the asset at a unit price of S_0. By the process of continuous compounding these become $Ne^{q\tau}$ units at time τ, which, according to the forwards contract, will sell at a unit price of $F_{0|\tau}$ yielding a cash flow of $NF_{0|\tau}e^{q\tau}$. The present value of this cash flow at time $t = 0$, discounted using the risk free rate of interest r, is $N\{F_{0|\tau}Ne^{q\tau}\}e^{-r\tau}$. This must equal the spot value, which is NS_0. Therefore, $S_0 = F_{0|\tau}e^{(q-r)\tau}$, which is equivalent to the stated condition.

The Present Value of a Fixed-Interest Bond. Let r be the current annual rate of interest paid on an fixed-interest bond that matures in n years when the principal is repaid, and let $\delta = (1 + r)^{-1}$ be the annual discount factor. Then, the present value of the income stream is equal the face value of the bond.

To show this, let A be be the face value of the bond, and let Q be the present value of the income stream. Then, Ar is the size of the annual interest payment, calculated as a fixed percentage of the face value, and the present value of the income stream is

$$Q = Ar\delta(1 + \delta + \delta^2 + \cdots + \delta^{n-1}) + A\delta^n.$$

We observe that

$$1 + \delta + \delta^2 + \cdots + \delta^{n-1} = \frac{1 - \delta^n}{1 - \delta}$$

and that

$$1 - \delta = r\delta.$$

Substituting these into the expression for the present value gives

$$Q = A(1 - \delta^n) + A\delta^n = A.$$