BLACK–SCHOLES OPTION PRICING

The Differential Equation The Black-Scholes model of option pricing assumes that the price S_t of the underlying asset has a geometric Brownian motion, which is to say that

$$dS = \mu S dt + \sigma S dw, \tag{1}$$

where μ and σ are constant parameters.

Let $f = f(S, t)$ be the price of a derivative, which might be a call option contingent on the price S of the underlying asset. Then, according to Ito's lemma, there is

$$df = \left\{ \mu S \frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \sigma^2 S^2 \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \right\} dt + \sigma S \frac{\partial f}{\partial S} dw. \tag{2}$$

Because the forcing function in both equations (1) and (2) is the same Wiener process, it is possible to construct a portfolio that eliminates risk, which is to say that the Wiener increment dw can be eliminated from the equation expressing the value of the portfolio.

The relevant portfolio comprises $\partial f/\partial S$ units of the asset and 1 unit of the derivative or option as a liability. The value of the portfolio at time t is given by

$$V_t = \frac{\partial f}{\partial S} S_t - f_t; \tag{3}$$

and the change in its value is

$$dV_t = \frac{\partial f}{\partial S} dS_t - df_t. \tag{4}$$

Substituting (1) and (2) into the latter and cancelling various terms gives

$$dV_t = -\left\{ \frac{\partial f}{\partial t} + \sigma^2 S^2 \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \right\} dt, \tag{5}$$

which does not involve the stochastic increment dw.

On the instant, this portfolio must earn the same as a sum V_t invested in a riskless asset at a rate of return of r, which is to say that

$$dV_t = r V_t dt. \tag{6}$$

By putting (5) into the LHS of this and (3) into the RHS, we get

$$\left\{ \frac{\partial f}{\partial t} + \sigma^2 S^2 \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \right\} dt = r \left\{ f_t - \frac{\partial f}{\partial S} S_t \right\} dt \tag{7}$$

whence

$$\frac{\partial f}{\partial t} + \sigma^2 S^2 \frac{1}{2} \frac{\partial^2 f}{\partial S^2} + r \frac{\partial f}{\partial S} dS_t = r f_t. \tag{8}$$

This it the Black-Scholes differential equation for option pricing.