LECTURE 4 : ARMA PROCESSES

Moving-Average Processes

The MA(q) process, is defined by

$$
\begin{aligned}
y(t) &= \mu_0\varepsilon(t) + \mu_1\varepsilon(t-1) + \cdots + \mu_q\varepsilon(t-q) \\
&= \mu(L)\varepsilon(t),
\end{aligned} \tag{5.3}
$$

where $\mu(L) = \mu_0 + \mu_1 L + \cdots + \mu_q L^q$ and where $\varepsilon(t)$ is white noise.

An MA model should be invertible such that $\mu^{-1}(L)y(t) = \varepsilon(t)$. This AR($\infty$) representation is is available if and only if all the roots of $\mu(z) = 0$ lie outside the unit circle.

Example. Consider the MA(1) process

$$
y(t) = \varepsilon(t) - \theta\varepsilon(t-1) = (1-\theta L)\varepsilon(t). \tag{5.4}
$$

Provided that $|\theta| < 1$, this can be written in autoregressive form as

$$
\begin{aligned}
\varepsilon(t) &= (1-\theta L)^{-1}y(t) \\
&= \big\{y(t) + \theta y(t-1) + \theta^2 y(t-2) + \cdots\big\}.
\end{aligned} \tag{5.5}
$$

Imagine that $|\theta| > 1$ instead. Then we have to write

$$
\begin{aligned}
y(t+1) &= \varepsilon(t+1) - \theta\varepsilon(t) \\
&= -\theta(1 - L^{-1}/\theta)\varepsilon(t),
\end{aligned} \tag{5.6}
$$

where $L^{-1}\varepsilon(t) = \varepsilon(t+1)$. This gives

$$
\begin{aligned}
\varepsilon(t) &= -\theta^{-1}(1 - L^{-1}/\theta)^{-1}y(t+1) \\
&= -\theta^{-1}\big\{y(t+1)/\theta + y(t+2)/\theta^2 + y(t-3)/\theta^3 + \cdots\big\}.
\end{aligned} \tag{5.7}
$$

Normally, an expression such as this, which embodies future values of $y(t)$, would have no reasonable meaning.

The Autocovariances of an MA Process

Consider

$$\begin{aligned}\gamma_\tau &= E(y_t y_{t-\tau})\\ &= E\Big\{\sum_i \mu_i \varepsilon_{t-i} \sum_j \mu_j \varepsilon_{t-\tau-j}\Big\}\\ &= \sum_i \sum_j \mu_i \mu_j E(\varepsilon_{t-i}\varepsilon_{t-\tau-j}).\end{aligned} \tag{5.8}$$

Since $\varepsilon(t)$ is white noise, it follows that

$$E(\varepsilon_{t-i}\varepsilon_{t-\tau-j}) = \begin{cases} 0, & \text{if } i \neq \tau + j;\\ \sigma_\varepsilon^2, & \text{if } i = \tau + j.\end{cases} \tag{5.9}$$

Therefore

$$\gamma_\tau = \sigma_\varepsilon^2 \sum_j \mu_j \mu_{j+\tau}. \tag{5.10}$$

Now let $\tau = 0, 1, \ldots, q$. This gives

$$\begin{aligned}\gamma_0 &= \sigma_\varepsilon^2(\mu_0^2 + \mu_1^2 + \cdots + \mu_q^2),\\ \gamma_1 &= \sigma_\varepsilon^2(\mu_0\mu_1 + \mu_1\mu_2 + \cdots + \mu_{q-1}\mu_q),\\ &\vdots\\ \gamma_q &= \sigma_\varepsilon^2 \mu_0 \mu_q.\end{aligned} \tag{5.11}$$

Also, $\gamma_\tau = 0$ for all $\tau > q$.

Example. The MA(1) process $y(t) = \varepsilon(t) - \theta\varepsilon(t-1)$ has

$$\begin{aligned}\gamma_0 &= \sigma_\varepsilon^2(1+\theta^2),\\ \gamma_1 &= -\sigma_\varepsilon^2\theta,\\ \gamma_\tau &= 0 \quad \text{if} \quad \tau > 1.\end{aligned} \tag{5.12}$$

Thus the dispersion matrix of $y = [y_1, y_2, \ldots, y_T]'$ is

$$D(y) = \sigma_\varepsilon^2 \begin{bmatrix} 1+\theta^2 & -\theta & 0 & \ldots & 0\\ -\theta & 1+\theta^2 & -\theta & \ldots & 0\\ 0 & -\theta & 1+\theta^2 & \ldots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \ldots & 1+\theta^2 \end{bmatrix}. \tag{5.13}$$

Autocovariance Generating Function

This is denoted by

$$\gamma(z) = \sum_\tau \gamma_\tau z^\tau; \quad \text{with} \quad \tau = \{0, \pm1, \pm2, \ldots\} \quad \text{and} \quad \gamma_\tau = \gamma_{-\tau}. \tag{5.14}$$

To find the autocovariance generating function of the MA(q) process, consider

$$\begin{aligned} \mu(z)\mu(z^{-1}) &= \sum_i \mu_i z^i \sum_j \mu_j z^{-j} \\ &= \sum_i \sum_j \mu_i \mu_j z^{i-j} \\ &= \sum_\tau \Big(\sum_j \mu_i \mu_{j+\tau} \Big) z^\tau, \quad \tau = i - j. \end{aligned} \tag{5.15}$$

From (10) it follows that

$$\gamma(z) = \sigma_\varepsilon^2 \mu(z)\mu(z^{-1}). \tag{5.16}$$

Autoregressive Processes

The AR(p) process, is defined by

$$\alpha_0 y(t) + \alpha_1 y(t-1) + \cdots + \alpha_p y(t-p) = \varepsilon(t). \tag{5.17}$$

This can be written as $\alpha(L)y(t) = \varepsilon(t)$, where $\alpha(L) = \alpha_0 + \alpha_1 L + \cdots + \alpha_p L^p$. For the process to be stationary, the roots of $\alpha(z) = 0$ must lie outside the unit circle. In that case the AR process can be written as an MA(∞) process: $y(t) = \alpha^{-1}(L)\varepsilon(t)$.

The autocovariance generating function for the AR(p) process is

$$\gamma(z) = \frac{\sigma_\varepsilon^2}{\alpha(z)\alpha(z^{-1})}. \tag{5.23}$$

Example. Consider the AR(1) process defined by

$$\begin{aligned} \varepsilon(t) &= y(t) - \phi y(t-1) \\ &= (1 - \phi L) y(t). \end{aligned} \tag{5.18}$$

Provided that $|\phi| < 1$, this can be represented in MA form as

$$\begin{aligned} y(t) &= (1-\phi L)^{-1}\varepsilon(t) \\ &= \big\{\varepsilon(t) + \phi\varepsilon(t-1) + \phi^2\varepsilon(t-2) + \cdots\big\}. \end{aligned} \tag{5.19}$$

The autocovariances of the AR(1) process can be obtained via the formula (10) for the autocovarainces of an MA process. Thus

$$\begin{aligned} \gamma_\tau &= E(y_t y_{t-\tau}) \\ &= E\Big\{\sum_i \phi^i \varepsilon_{t-i} \sum_j \phi^j \varepsilon_{t-\tau-j}\Big\} \\ &= \sum_i \sum_j \phi^i \phi^j E(\varepsilon_{t-i}\varepsilon_{t-\tau-j}); \end{aligned} \tag{5.20}$$

and it follows from (9) that

$$\begin{aligned} \gamma_\tau &= \sigma_\varepsilon^2 \sum_j \phi^j \phi^{j+\tau} \\ &= \frac{\sigma_\varepsilon^2 \phi^\tau}{1-\phi^2}. \end{aligned} \tag{5.21}$$

The dispersion matrix of $y = [y_1, y_2, \ldots, y_T]'$ is

$$D(y) = \frac{\sigma_\varepsilon^2}{1-\phi^2}\begin{bmatrix} 1 & \phi & \phi^2 & \ldots & \phi^{T-1} \\ \phi & 1 & \phi & \ldots & \phi^{T-2} \\ \phi^2 & \phi & 1 & \ldots & \phi^{T-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi^{T-1} & \phi^{T-2} & \phi^{T-3} & \ldots & 1 \end{bmatrix}. \tag{5.22}$$

The Yule-Walker Equations

For an alternative way of finding the AR autocovariances, consider multiplying $\sum_i \alpha_i y_{t-i} = \varepsilon_t$ by $y_{t-\tau}$ and taking expectations to give

$$\sum_i \alpha_i E(y_{t-i}y_{t-\tau}) = E(\varepsilon_t y_{t-\tau}). \tag{5.24}$$

Given that $\alpha_0 = 1$, it follows that

$$E(\varepsilon_t y_{t-\tau}) = \begin{cases} \sigma_\varepsilon^2, & \text{if } \tau = 0; \\ 0, & \text{if } \tau > 0. \end{cases} \tag{5.25}$$

Therefore, on setting $E(y_{t-i}y_{t-\tau}) = \gamma_{\tau-i}$, equation (24) gives

$$\sum_i \alpha_i \gamma_{\tau-i} = \begin{cases} \sigma_\varepsilon^2, & \text{if } \tau = 0; \\ 0, & \text{if } \tau > 0. \end{cases} \tag{5.26}$$

The second of these is a homogeneous difference equation which enables us to generate the sequence $\{\gamma_p, \gamma_{p+1}, \ldots\}$ once p starting values $\gamma_0, \gamma_1, \ldots, \gamma_{p-1}$ are known. By letting $\tau = 0, 1, \ldots, p$ in (26), we generate a set of $p+1$ equations which can be arrayed in matrix form as follows:

$$\begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_p \\ \gamma_1 & \gamma_0 & \gamma_1 & \cdots & \gamma_{p-1} \\ \gamma_2 & \gamma_1 & \gamma_0 & \cdots & \gamma_{p-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_p & \gamma_{p-1} & \gamma_{p-2} & \cdots & \gamma_0 \end{bmatrix} \begin{bmatrix} 1 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_p \end{bmatrix} = \begin{bmatrix} \sigma_\varepsilon^2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{5.27}$$

These are called the Yule–Walker equations, and they can be used either for generating the values $\gamma_0, \gamma_1, \ldots, \gamma_p$ from the values $\alpha_1, \ldots, \alpha_p, \sigma_\varepsilon^2$ or vice versa.

Example. Consider the second-order autoregressive process. We have

$$\begin{aligned} \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 \\ \gamma_1 & \gamma_0 & \gamma_1 \\ \gamma_2 & \gamma_1 & \gamma_0 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} &= \begin{bmatrix} \alpha_2 & \alpha_1 & \alpha_0 & 0 & 0 \\ 0 & \alpha_2 & \alpha_1 & \alpha_0 & 0 \\ 0 & 0 & \alpha_2 & \alpha_1 & \alpha_0 \end{bmatrix} \begin{bmatrix} \gamma_2 \\ \gamma_1 \\ \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{bmatrix} \\ &= \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 \\ \alpha_1 & \alpha_0 + \alpha_2 & 0 \\ \alpha_2 & \alpha_1 & \alpha_0 \end{bmatrix} \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} \sigma_\varepsilon^2 \\ 0 \\ 0 \end{bmatrix}. \end{aligned} \tag{5.28}$$

Given $\alpha_0 = 1$ and the values for $\gamma_0, \gamma_1, \gamma_2$, we can find σ_ε^2 and α_1, α_2. Conversely, given $\alpha_0, \alpha_1, \alpha_2$ and σ_ε^2, we can find $\gamma_0, \gamma_1, \gamma_2$.

The Partial Autocorrelation Function

Let $\alpha_{r(r)}$ be the coefficient associated with $y(t-r)$ in an autoregressive process of order r whose parameters correspond to the autocovariances $\gamma_0, \gamma_1, \ldots, \gamma_r$. Then the sequence $\{\alpha_{r(r)}; r = 1, 2, \ldots\}$ of such coefficients, whose index corresponds to models of increasing orders, constitutes the partial autocorrelation function. In effect, $\alpha_{r(r)}$ indicates the role in explaining the variance of $y(t)$ which is due to $y(t-r)$ when $y(t-1), \ldots, y(t-r+1)$ are also taken into account.

The sequence of partial autocorrelations may be computed efficiently via the recursive Durbin–Levinson Algorithm which uses the coefficients of the AR model of order r as the basis for calculating the coefficients of the model of order $r+1$.

Imagine that we already have the values $\alpha_{0(r)} = 1, \alpha_{1(r)}, \ldots, \alpha_{r(r)}$. Then, by extending the set of rth-order Yule–Walker equations to which these values correspond, we can derive the system

$$\begin{bmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_r & \gamma_{r+1} \\ \gamma_1 & \gamma_0 & \cdots & \gamma_{r-1} & \gamma_r \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma_r & \gamma_{r-1} & \cdots & \gamma_0 & \gamma_1 \\ \gamma_{r+1} & \gamma_r & \cdots & \gamma_1 & \gamma_0 \end{bmatrix} \begin{bmatrix} 1 \\ \alpha_{1(r)} \\ \vdots \\ \alpha_{r(r)} \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma^2_{(r)} \\ 0 \\ \vdots \\ 0 \\ g \end{bmatrix}, \tag{5.29}$$

wherein

$$g = \sum_{j=0}^{r} \alpha_{j(r)} \gamma_{r+1-j} \qquad \text{with} \quad \alpha_{0(r)} = 1. \tag{5.30}$$

The system can also be written as

$$\begin{bmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_r & \gamma_{r+1} \\ \gamma_1 & \gamma_0 & \cdots & \gamma_{r-1} & \gamma_r \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma_r & \gamma_{r-1} & \cdots & \gamma_0 & \gamma_1 \\ \gamma_{r+1} & \gamma_r & \cdots & \gamma_1 & \gamma_0 \end{bmatrix} \begin{bmatrix} 0 \\ \alpha_{r(r)} \\ \vdots \\ \alpha_{1(r)} \\ 1 \end{bmatrix} = \begin{bmatrix} g \\ 0 \\ \vdots \\ 0 \\ \sigma^2_{(r)} \end{bmatrix}. \tag{5.31}$$

The two systems of equations (29) and (31) can be combined to give

$$\begin{bmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_r & \gamma_{r+1} \\ \gamma_1 & \gamma_0 & \cdots & \gamma_{r-1} & \gamma_r \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma_r & \gamma_{r-1} & \cdots & \gamma_0 & \gamma_1 \\ \gamma_{r+1} & \gamma_r & \cdots & \gamma_1 & \gamma_0 \end{bmatrix} \begin{bmatrix} 1 \\ \alpha_{1(r)} + c\alpha_{r(r)} \\ \vdots \\ \alpha_{r(r)} + c\alpha_{1(r)} \\ c \end{bmatrix} = \begin{bmatrix} \sigma^2_{(r)} + cg \\ 0 \\ \vdots \\ 0 \\ g + c\sigma^2_{(r)} \end{bmatrix}. \tag{5.32}$$

If we take the coefficient of the combination to be

$$c = -\frac{g}{\sigma^2_{(r)}}, \tag{5.33}$$

then the final element in the vector on the RHS becomes zero and the system becomes the set of Yule–Walker equations of order $r+1$. The solution of the equations, from the last element $\alpha_{r+1(r+1)} = c$ through to the variance term $\sigma^2_{(r+1)}$ is given by

$$\alpha_{r+1(r+1)} = \frac{1}{\sigma^2_{(r)}}\left\{\sum_{j=0}^{r}\alpha_{j(r)}\gamma_{r+1-j}\right\}$$

$$\begin{bmatrix}\alpha_{1(r+1)}\\ \vdots\\ \alpha_{r(r+1)}\end{bmatrix} = \begin{bmatrix}\alpha_{1(r)}\\ \vdots\\ \alpha_{r(r)}\end{bmatrix} + \alpha_{r+1(r+1)}\begin{bmatrix}\alpha_{r(r)}\\ \vdots\\ \alpha_{1(r)}\end{bmatrix} \tag{5.34}$$

$$\sigma^2_{(r+1)} = \sigma^2_{(r)}\big\{1-(\alpha_{r+1(r+1)})^2\big\}.$$

Thus the solution of the Yule–Walker system of order $r+1$ is easily derived from the solution of the system of order r, and there is scope for devising a recursive procedure. The starting values for the recursion are

$$\alpha_{1(1)} = -\gamma_1/\gamma_0 \quad \text{and} \quad \sigma^2_{(1)} = \gamma_0\big\{1-(\alpha_{1(1)})^2\big\}. \tag{5.35}$$

Autoregressive Moving Average Processes

The ARMA(p,q) process, is defined by

$$
\begin{aligned}
(5.36)\qquad & \alpha_0 y(t) + \alpha_1 y(t-1) + \cdots + \alpha_p y(t-p) \\
& \quad = \mu_0 \varepsilon(t) + \mu_1 \varepsilon(t-1) + \cdots + \mu_q \varepsilon(t-q).
\end{aligned}
$$

This can also be written as $\alpha(L)y(t) = \mu(L)\varepsilon(t)$. If the roots of $\alpha(z) = 0$ lie outside the unit circle, then the process has an MA(∞) form: $y(t) = \alpha^{-1}(L)\mu(L)\varepsilon(t)$. If the roots of $\mu(z) = 0$ lie outside the unit circle, then it has an AR(∞) form: $\mu^{-1}(L)\alpha(L)y(t) = \varepsilon(t)$.

The autocovariance generating function for the ARMA process is

$$
(5.37)\qquad \gamma(z) = \sigma_\varepsilon^2 \frac{\mu(z)\mu(z^{-1})}{\alpha(z)\alpha(z^{-1})}.
$$

To find the autocovariances in practice, consider multiplying the equation $\sum_i \alpha_i y_{t-i} = \sum_i \mu_i \varepsilon_{t-i}$ by $y_{t-\tau}$ and taking expectations. This gives

$$
(5.38)\qquad \sum_i \alpha_i \gamma_{\tau-i} = \sum_i \mu_i \delta_{i-\tau},
$$

where $\gamma_{\tau-i} = E(y_{t-\tau} y_{t-i})$ and $\delta_{i-\tau} = E(y_{t-\tau}\varepsilon_{t-i})$. Since ε_{t-i} is uncorrelated with $y_{t-\tau}$ whenever it is subsequent to the latter, it follows that $\delta_{i-\tau} = 0$ if $\tau > i$. Since the index i in the RHS of the equation (38) runs from 0 to q, it follows that

$$
(5.39)\qquad \sum_i \alpha_i \gamma_{i-\tau} = 0 \quad \text{if} \quad \tau > q.
$$

Given the $q+1$ nonzero values $\delta_0, \delta_1, \ldots, \delta_q$, and p initial values $\gamma_0, \gamma_1, \ldots, \gamma_{p-1}$, the equations can be solved recursively for $\{\gamma_p, \gamma_{p+1}, \ldots\}$.

To find the requisite values $\delta_0, \delta_1, \ldots, \delta_q$, consider multiplying the equation $\sum_i \alpha_i y_{t-i} = \sum_i \mu_i \varepsilon_{t-i}$ by $\varepsilon_{t-\tau}$ and taking expectations. This gives

$$
(5.40)\qquad \sum_i \alpha_i \delta_{\tau-i} = \mu_\tau \sigma_\varepsilon^2,
$$

where $\delta_{\tau-i} = E(y_{t-i}\varepsilon_{t-\tau})$. The equation may be rewritten as

$$
(5.41)\qquad \delta_\tau = \frac{1}{\alpha_0}\Big(\mu_\tau \sigma_\varepsilon^2 - \sum_{i=1} \delta_{\tau-i}\Big),
$$

and, by setting $\tau = 0, 1, \ldots, q$, we can generate recursively the required values $\delta_0, \delta_1, \ldots, \delta_q$.

Example. Consider the ARMA(2, 2) model which gives the equation

$$\alpha_0 y_t + \alpha_1 y_{t-1} + \alpha_2 y_{t-2} = \mu_0 \varepsilon_t + \mu_1 \varepsilon_{t-1} + \mu_2 \varepsilon_{t-2}. \tag{5.42}$$

Multiplying by y_t, y_{t-1} and y_{t-2} and taking expectations gives

$$\begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 \\ \gamma_1 & \gamma_0 & \gamma_1 \\ \gamma_2 & \gamma_1 & \gamma_0 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \delta_0 & \delta_1 & \delta_2 \\ 0 & \delta_0 & \delta_1 \\ 0 & 0 & \delta_0 \end{bmatrix} \begin{bmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \end{bmatrix}. \tag{5.43}$$

Multiplying by ε_t, ε_{t-1} and ε_{t-2} and taking expectations gives

$$\begin{bmatrix} \delta_0 & 0 & 0 \\ \delta_1 & \delta_0 & 0 \\ \delta_2 & \delta_1 & \delta_0 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \sigma_\varepsilon^2 & 0 & 0 \\ 0 & \sigma_\varepsilon^2 & 0 \\ 0 & 0 & \sigma_\varepsilon^2 \end{bmatrix} \begin{bmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \end{bmatrix}. \tag{5.44}$$

When the latter equations are written as

$$\begin{bmatrix} \alpha_0 & 0 & 0 \\ \alpha_1 & \alpha_0 & 0 \\ \alpha_2 & \alpha_1 & \alpha_0 \end{bmatrix} \begin{bmatrix} \delta_0 \\ \delta_1 \\ \delta_2 \end{bmatrix} = \sigma_\varepsilon^2 \begin{bmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \end{bmatrix}, \tag{5.45}$$

they can be solved recursively for δ_0, δ_1 and δ_2 on the assumption that that the values of α_0, α_1, α_2 and σ_ε^2 are known. Notice that, when we adopt the normalisation $\alpha_0 = \mu_0 = 1$, we get $\delta_0 = \sigma_\varepsilon^2$. When the equations (43) are rewritten as

$$\begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 \\ \alpha_1 & \alpha_0 + \alpha_2 & 0 \\ \alpha_2 & \alpha_1 & \alpha_0 \end{bmatrix} \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} \mu_0 & \mu_1 & \mu_2 \\ \mu_1 & \mu_2 & 0 \\ \mu_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_0 \\ \delta_1 \\ \delta_2 \end{bmatrix}, \tag{5.46}$$

they can be solved for γ_0, γ_1 and γ_2. Thus the starting values are obtained which enable the equation

$$\alpha_0 \gamma_\tau + \alpha_1 \gamma_{\tau-1} + \alpha_2 \gamma_{\tau-2} = 0; \quad \tau > 2 \tag{5.47}$$

to be solved recursively to generate the succeeding values $\{\gamma_3, \gamma_4, \ldots\}$ of the autocovariances.