LECTURE 2 : MODELS AND METHODS

Time-Series Models: Feedback Form and Transfer-Function Form

A dynamic regression model is a relationship comprising any number of consecutive elements of $x(t)$, $y(t)$ and $\varepsilon(t)$:

$$\sum_{i=0}^{p}\alpha_i y(t-i)=\sum_{i=0}^{k}\beta_i x(t-i)+\sum_{i=0}^{q}\mu_i\varepsilon(t-i), \tag{3.2}$$

It is taken for granted that $\alpha_0=1$.

A time-series model can assume a variety of forms. Consider a simple dynamic regression model

$$y(t)=\phi y(t-1)+x(t)\beta+\varepsilon(t), \tag{3.5}$$

By repeated substitution, we obtain

$$\begin{aligned}y(t)&=\phi y(t-1)+\beta x(t)+\varepsilon(t)\\&=\phi^2 y(t-2)+\beta\big\{x(t)+\phi x(t-1)\big\}+\varepsilon(t)+\phi\varepsilon(t-1)\\&\;\;\vdots\\&=\phi^n y(t-n)+\beta\big\{x(t)+\phi x(t-1)+\cdots+\phi^{n-1}x(t-n+1)\big\}\\&\qquad+\varepsilon(t)+\phi\varepsilon(t-1)+\cdots+\phi^{n-1}\varepsilon(t-n+1).\end{aligned} \tag{3.6}$$

If $|\phi|<1$, then $\lim(n\to\infty)\phi^n=0$; and the limiting form is

$$y(t)=\beta\sum_{i=0}^{\infty}\phi^i x(t-i)+\sum_{i=0}^{\infty}\phi^i\varepsilon(t-i). \tag{3.7}$$

The Lag Operator

We can define polynomials of the lag operator of the form $p(L)=p_0+p_1L+\cdots+p_nL^n=\sum p_iL^i$ having the effect that

$$\begin{aligned}p(L)x(t)&=p_0x(t)+p_1x(t-1)+\cdots+p_nx(t-n)\\&=\sum_{i=0}^{n}p_ix(t-i).\end{aligned} \tag{3.11}$$

In these terms, the equation under (2) becomes

$$\alpha(L)y(t)=\beta(L)x(t)+\mu(L)\varepsilon(t). \tag{3.12}$$

Complex Numbers: Euler's Equations

There are three alternative ways of representing the conjugate complex numbers λ and λ^* :

$$\begin{aligned} \lambda &= \alpha + i\beta = \rho(\cos\theta + i\sin\theta) = \rho e^{i\theta}, \\ \lambda^* &= \alpha - i\beta = \rho(\cos\theta - i\sin\theta) = \rho e^{-i\theta}, \end{aligned} \tag{3.14}$$

where

$$\rho = \sqrt{\alpha^2 + \beta^2} \qquad \text{and} \qquad \theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right). \tag{3.15}$$

These are the Cartesian form, the trigonometrical form and the exponential form.

To understand the exponential form consider the series expansions of $\cos\theta$ and $i\sin\theta$ about the point $\theta = 0$:

$$\begin{aligned} \cos\theta &= \left\{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right\}, \\ i\sin\theta &= \left\{i\theta - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} - \frac{i\theta^7}{7!} + \cdots\right\}. \end{aligned} \tag{3.16}$$

Adding these gives

$$\begin{aligned} \cos\theta + i\sin\theta &= \left\{1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots\right\} \\ &= e^{i\theta}. \end{aligned} \tag{3.17}$$

Likewise, by subtraction, we get

$$\begin{aligned} \cos\theta - i\sin\theta &= \left\{1 - i\theta - \frac{\theta^2}{2!} + \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} - \cdots\right\} \\ &= e^{-i\theta}. \end{aligned} \tag{3.18}$$

These are Euler's equations. It follows from adding (17) and (18) that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}. \tag{3.19}$$

Subtracting (18) from (17) gives

$$\begin{aligned} \sin\theta &= \frac{-i}{2}(e^{i\theta} - e^{-i\theta}) \\ &= \frac{1}{2i}(e^{i\theta} - e^{-i\theta}). \end{aligned} \tag{3.20}$$

The Polynomial Equation of Order n.

Now consider the general equation of the nth order:

$$\phi_0 + \phi_1 z + \phi_2 z^2 + \cdots + \phi_n z^n = 0. \tag{3.21}$$

On dividing by ϕ_n, we can factorise this as

$$(z - \lambda_1)(z - \lambda_2)\cdots(z - \lambda_n) = 0. \tag{3.22}$$

When we multiply the n factors together, we obtain the expansion

$$0 = z^n - \sum_i \lambda_i z^{n-1} + \sum_i \sum_j \lambda_i \lambda_j z^{n-2} - \cdots (-1)^n \lambda_1 \lambda_2 \cdots \lambda_n. \tag{3.23}$$

This is compared with the expression $(\phi_0/\phi_n) + (\phi_1/\phi_n)z + \cdots + z^n = 0$. By equating coefficients of the two expressions, we find that $(\phi_0/\phi_n) = (-1)^n \prod \lambda_i$ or, equivalently,

$$\phi_n = \phi_0 \prod_{i=1}^{n} (-\lambda_i)^{-1}. \tag{3.24}$$

Thus we can express the polynomial in any of the following forms:

$$\begin{aligned} \sum \phi_i z^i &= \phi_n \prod (z - \lambda_i) \\ &= \phi_0 \prod (-\lambda_i)^{-1} \prod (z - \lambda_i) \\ &= \phi_0 \prod \left(1 - \frac{z}{\lambda_i}\right). \end{aligned} \tag{3.25}$$

If λ is a root of the primary equation

$$\phi(z) = \phi_0 + \phi_1 z + \cdots \phi_n z^n = 0$$

where rising powers of z are associated with rising indices on the coefficients, then $\mu = 1/\lambda$ is a root of the auxilliary equation

$$\phi'(z) = z^n \phi(z^{-1}) = \phi_0 z^n + \phi_1 z^{n-1} + \cdots \phi_n = 0$$

which has declining powers of z instead.

Rational Functions of Polynomials

(3.27) If $\delta(z)/\gamma(z) = \delta(z)/\{\gamma_1(z)\gamma_2(z)\}$ is a proper rational function, and if $\gamma_1(z)$ and $\gamma_2(z)$ have no common factor, then the function can be uniquely expressed as

$$\frac{\delta(z)}{\gamma(z)} = \frac{\delta_1(z)}{\gamma_1(z)} + \frac{\delta_2(z)}{\gamma_2(z)},$$

where $\delta_1(z)/\gamma_1(z)$ and $\delta_2(z)/\gamma_2(z)$ are proper rational functions.

Imagine that $\gamma(z) = \prod(1-z/\lambda_i)$. Then repeated applications of this result enables us to write

$$\frac{\delta(z)}{\gamma(z)} = \frac{\kappa_1}{1 - z/\lambda_1} + \frac{\kappa_2}{1 - z/\lambda_2} + \cdots + \frac{\kappa_n}{1 - z/\lambda_n}. \tag{3.28}$$

By adding the terms on the RHS, we find an expression with a numerator of degree $n-1$. By equating the terms of the numerator with the terms of $\delta(z)$, we can find the values $\kappa_1, \kappa_2, \ldots, \kappa_n$. The series expansion of $\delta(z)/\gamma(z)$converges if and only if the expansion of each of the partial fractions converges. For

$$\frac{\kappa}{1 - z/\lambda} = \kappa\Big\{1 + z/\lambda + (z/\lambda)^2 + \cdots\Big\} \tag{3.29}$$

to converge when $|z| \leq 1$, it is necessary and sufficient that $|\lambda| > 1$.

Linear Difference Equations

An nth-order linear difference equation is a relationship amongst $n+1$ consecutive elements of a sequence $x(t)$ of the form

$$\alpha_0 x(t) + \alpha_1 x(t-1) + \cdots + \alpha_n x(t-n) = u(t), \tag{3.32}$$

where $u(t)$ is some specified sequence which is described as the forcing function. The equation can be written, in a summary notation, as

$$\alpha(L)x(t) = u(t), \tag{3.33}$$

where $\alpha(L) = \alpha_0 + \alpha_1 L + \cdots + \alpha_n L^n$.

Solution of the Homogeneous Difference Equation

If λ_j is a root of the equation $\alpha(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_n z^n = 0$ such that $\alpha(\lambda_j) = 0$, then $y_j(t) = (1/\lambda_j)^t$ is a solution of the equation $\alpha(L)y(t) = 0$. Thus

$$\begin{aligned}
\alpha(L)\left(\frac{1}{\lambda_j}\right)^t &= \left(\alpha_0 + \alpha_1 L + \cdots + \alpha_n L^n\right)\left(\frac{1}{\lambda_j}\right)^t \\
&= \alpha_0\left(\frac{1}{\lambda_j}\right)^t + \alpha_1\left(\frac{1}{\lambda_j}\right)^{t-1} + \cdots + \alpha_n\left(\frac{1}{\lambda_j}\right)^{t-n} \\
&= \left(\alpha_0 + \alpha_1\lambda_j + \cdots + \alpha_n\lambda_j^n\right)\left(\frac{1}{\lambda_j}\right)^t \\
&= \alpha(\lambda_j)\left(\frac{1}{\lambda_j}\right)^t = 0.
\end{aligned} \tag{3.34}$$

Also consider the factorisation $\alpha(L) = \alpha_0\prod_i(1 - L/\lambda_i)$. Within this product is the term $1 - L/\lambda_j$; and since

$$\left(1 - \frac{L}{\lambda_j}\right)\left(\frac{1}{\lambda_j}\right)^t = \left(\frac{1}{\lambda_j}\right)^t - \left(\frac{1}{\lambda_j}\right)^t = 0,$$

it follows that $\alpha(L)(1/\lambda_j)^t = 0$.

The general solution, in the case where $\alpha(L) = 0$ has distinct real roots, is given by

$$y(t;c) = c_1\left(\frac{1}{\lambda_1}\right)^t + c_2\left(\frac{1}{\lambda_2}\right)^t + \cdots + c_n\left(\frac{1}{\lambda_n}\right)^t, \tag{3.35}$$

where $c_1, c_2, \ldots, c_n$ are the constants which are determined by the initial conditions.

Solution of Difference Equations: The Case of Repeated Roots

If two roots coincide at a value of λ_j, then $\alpha(L)y(t) = 0$ has the solutions $y_1(t) = (1/\lambda_j)^t$ and $y_2(t) = t(1/\lambda_j)^t$.

To show that $y_2(t)$ is also a solution, consider

$$
\begin{aligned}
\left(1 - \frac{L}{\lambda_j}\right)^2 t\left(\frac{1}{\lambda_j}\right)^t &= \left(1 - \frac{2L}{\lambda_j} + \frac{L^2}{\lambda_j^2}\right) t\left(\frac{1}{\lambda_j}\right)^t \\
&= t\left(\frac{1}{\lambda_j}\right)^t - 2(t-1)\left(\frac{1}{\lambda_j}\right)^t + (t-2)\left(\frac{1}{\lambda_j}\right)^t = 0.
\end{aligned} \tag{3.36}
$$

If there are r repeated roots with a value of λ_j, then all of $(1/\lambda_j)^t$, $t(1/\lambda_j)^t$, $t^2(1/\lambda_j)^t, \ldots, t^{r-1}(1/\lambda_j)^t$ are solutions, If there are r repeated roots of unit value, then the functions $1, t, t^2, \ldots, t^{r-1}$ are all solutions. Therefore the general solution of the homogeneous equation will contain a polynomial in t of degree $r - 1$:

$$d_0 + d_1 t + d_2 t^2 + \cdots + d_{r+1} t^{r+1}.$$

The 2nd-order Difference Equation with Complex Roots

Let $\alpha(L)y(t) = \alpha_0 y(t) + \alpha_1 y(t-1) + \alpha_2 y(t-2) = 0$ and suppose that $\alpha(z) = 0$ has complex roots $\lambda = 1/\mu$ and $\lambda^* = 1/\mu^*$. If λ, λ^* are conjugate complex numbers, then so too are μ, μ^*. Therefore

$$
\begin{aligned}
\mu &= \gamma + i\delta = \kappa(\cos\omega + i\sin\omega) = \kappa e^{i\omega}, \\
\mu^* &= \gamma - i\delta = \kappa(\cos\omega - i\sin\omega) = \kappa e^{-i\omega}.
\end{aligned} \tag{3.37}
$$

These will appear in a general solution of the difference equation of the form

$$y(t) = c\mu^t + c^*(\mu^*)^t. \tag{3.38}$$

This represents a real-valued sequence; and, since a real term must equal its own conjugate, it follows that c and c^* must be conjugate numbers of the form

$$
\begin{aligned}
c^* &= \rho(\cos\theta + i\sin\theta) = \rho e^{i\theta}, \\
c &= \rho(\cos\theta - i\sin\theta) = \rho e^{-i\theta}.
\end{aligned} \tag{3.39}
$$

Thus the general solution becomes

$$
\begin{aligned}
c\mu^t + c^*(\mu^*)^t &= \rho e^{-i\theta}(\kappa e^{i\omega})^t + \rho e^{i\theta}(\kappa e^{-i\omega})^t \\
&= \rho\kappa^t \left\{ e^{i(\omega t - \theta)} + e^{-i(\omega t - \theta)} \right\} \\
&= 2\rho\kappa^t \cos(\omega t - \theta).
\end{aligned} \tag{3.40}
$$

Transfer Functions

Consider the equation

$$\alpha(L)y(t) = \beta(L)x(t) + \varepsilon(t), \tag{3.65}$$

where

$$\begin{aligned}\alpha(L) &= 1 + \alpha_1 L + \cdots + \alpha_p L^p \\ &= 1 - \phi_1 L - \cdots - \phi_p L^p, \\ \beta(L) &= 1 + \beta_1 L + \cdots + \beta_k L^k\end{aligned} \tag{3.66}$$

are polynomials of the lag operator. The transfer-function form of the model is simply

$$y(t) = \frac{\beta(L)}{\alpha(L)}x(t) + \frac{1}{\alpha(L)}\varepsilon(t), \tag{3.67}$$

The rational function associated with $x(t)$ has a series expansion

$$\begin{aligned}\frac{\beta(z)}{\alpha(z)} &= \omega(z) \\ &= \left\{\omega_0 + \omega_1 z + \omega_2 z^2 + \cdots\right\};\end{aligned} \tag{3.68}$$

and the sequence of the coefficients of this expansion constitutes the impulse-response function. The partial sums of the coefficients constitute the step-response function. The gain of the transfer function is defined by

$$\gamma = \frac{\beta(1)}{\alpha(1)} = \frac{\beta_0 + \beta_1 + \cdots + \beta_k}{1 + \alpha_1 + \cdots + \alpha_p}. \tag{3.69}$$

Finding the Coefficients of the Impulse Response

Consider the second-order case:

$$\frac{\beta_0 + \beta_1 z}{1 - \phi_1 z - \phi_2 z^2} = \left\{\omega_0 + \omega_1 z + \omega_2 z^2 + \cdots\right\}. \tag{3.70}$$

We rewrite this equation as

$$\beta_0 + \beta_1 z = \left\{1 - \phi_1 z - \phi_2 z^2\right\}\left\{\omega_0 + \omega_1 z + \omega_2 z^2 + \cdots\right\}. \tag{3.71}$$

Then, by performing the multiplication on the RHS, and by equating the coefficients of the same powers of z on the two sides of the equation, we find that

$$\begin{aligned} \beta_0 &= \omega_0, & \omega_0 &= \beta_0, \\ \beta_1 &= \omega_1 - \phi_1\omega_0, & \omega_1 &= \beta_1 + \phi_1\omega_0, \\ 0 &= \omega_2 - \phi_1\omega_1 - \phi_2\omega_0, & \omega_2 &= \phi_1\omega_1 + \phi_2\omega_0, \\ &\vdots & &\vdots \\ 0 &= \omega_n - \phi_1\omega_{n-1} - \phi_2\omega_{n-2}, & \omega_n &= \phi_1\omega_{n-1} + \phi_2\omega_{n-2}. \end{aligned} \tag{3.72}$$

The coefficents are genereated by a homogeneous second-order difference equation:

$$\omega(n) = \phi_1\omega(n-1) + \phi_2\omega(n-2), \tag{3.73}$$

The Frequency Response

Consider mapping the signal $x(t) = \cos(\omega t)$ through the transfer function $\gamma(L) = \gamma_0 + \gamma_1 L + \cdots + \gamma_g L^g$. The output is

$$\begin{aligned} y(t) &= \gamma(L)\cos(\omega_t) \\ &= \sum_{j=0}^{g} \gamma_j \cos\big(\omega[t-j]\big). \end{aligned} \tag{3.74}$$

The trigonometrical identity $\cos(A - B) = \cos A \cos B + \sin A \sin B$ enables us to write this as

$$\begin{aligned} y(t) &= \Big\{\sum_j \gamma_j \cos(\omega j)\Big\}\cos(\omega t) + \Big\{\sum_j \gamma_j \sin(\omega j)\Big\}\sin(\omega t) \\ &= \alpha\cos(\omega t) + \beta\sin(\omega t) = \rho\cos(\omega t - \theta). \end{aligned} \tag{3.75}$$

Here we have defined

$$\begin{aligned} &\alpha = \sum_{j=0}^{g} \gamma_j \cos(\omega j), \qquad \beta = \sum_{j=0}^{g} \gamma_j \sin(\omega j), \\ &\rho = \sqrt{\alpha^2 + \beta^2} \qquad \text{and} \qquad \theta = \tan^{-1}\Big(\frac{\beta}{\alpha}\Big). \end{aligned} \tag{3.76}$$

It can be seen from (75) that the effect of the filter upon the signal is twofold. First there is a *gain effect* whereby the amplitude of the sinusoid has been increased or diminished by a factor of ρ. Also there is a *phase effect* whereby the peak of the sinusoid is displaced by a time delay of θ/ω periods. Figures 3 and 4 represent the two effects of a simple rational transfer function on the set of sinusoids whose frequencies range from 0 to π.