EG1110 Signals and Systems

Exercise 2: Convolution

1. Consider a discrete-time system $G$ which has impulse response

$$h(k) = \delta(k) + 2\delta(k - 1)$$

Use the convolution formula

$$y(k) = h(k) \ast u(k) = \sum_{n=-\infty}^{\infty} u(n)h(k - n)$$

to calculate the output $y(k)$ when the input is defined as

$$u(k) = \begin{cases} 2 & k = 1 \\ 3 & k = 2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Why will the output be zero (assuming zero initial conditions)

(a) before time $k = 1$; and

(b) after time $k = 3$?

2. Using the impulse response of $G$ given above, prove by direct calculation that if

$$y_1(k) = G[u_1(k)]$$
$$y_2(k) = G[u_2(k)]$$
$$u_1(k) = \begin{cases} 2 & k = 1 \\ 0 & \text{otherwise} \end{cases}$$
$$u_2(k) = \begin{cases} 3 & k = 2 \\ 0 & \text{otherwise} \end{cases}$$

then $y_1(k) + y_2(k) = y(k)$.

3. For continuous functions prove that

$$h(t) \ast u(t) = u(t) \ast h(t)$$

[Hint: using a change of variables $\lambda = t - \tau$, prove that

$$\int_{-\infty}^{\infty} u(\tau)h(t - \tau)d\tau = \int_{-\infty}^{\infty} y(t - \lambda)h(\lambda)d\lambda$$
]

4. For continuous functions prove that

$$x(t) \ast [y(t) \ast z(t)] = [x(t) \ast y(t)] \ast z(t)$$

[Hint: prove that both the left and right hand sides of the equation can be written as the double integral

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau) y(\lambda) z(t - \tau - \lambda)d\lambda d\tau$$

using appropriate changes of dummy variables.]
Solutions

1. The solution is a direct, if tedious, application of the convolution formula given in the question. First note that if

\[ h(k) = \delta(k) + 2\delta(k - 1) \]

then, replacing \( k \) with \( k - n \), it follows that

\[ h(k - n) = \delta(k - n) + 2\delta(k - n - 1) \]

So then we have

\[ y(k) = \sum_{n=-\infty}^{\infty} [\delta(k - n) + 2\delta(k - n - 1)]u(n) \]

Evaluating this for \( k = 0, 1, 2, 3 \) thus gives:

\[ y(0) = \sum_{n=-\infty}^{\infty} [\delta(-n) + 2\delta(-n - 1)]u(n) = [\delta(0) + 2\delta(-1)]u(0) = 0 \]

\[ y(1) = \sum_{n=-\infty}^{\infty} [\delta(1 - n) + 2\delta(1 - n - 1)]u(n) = [\delta(1 - 0) + 2\delta(1 - 1)]u(0) + [\delta(1 - 1) + 2\delta(1 - 1 - 1)]u(1) = \delta(0)u(1) = 1 \times 2 = 2 \]

\[ y(2) = \sum_{n=-\infty}^{\infty} [\delta(2 - n) + 2\delta(2 - n - 1)]u(n) = [\delta(2 - 0) + 2\delta(2 - 1)]u(0) + [\delta(2 - 1) + 2\delta(2 - 1 - 1)]u(1) + [\delta(2 - 2) + 2\delta(2 - 2 - 1)]u(2) = 2\delta(0)u(1) + \delta(0)u(2) = 2 \times 1 \times 2 + 1 \times 3 = 7 \]

\[ y(3) = \sum_{n=-\infty}^{\infty} [\delta(3 - n) + 2\delta(3 - n - 1)]u(n) = [\delta(3 - 0) + 2\delta(3 - 1)]u(0) + [\delta(3 - 1) + 2\delta(3 - 1 - 1)]u(1) + [\delta(3 - 2) + 2\delta(3 - 2 - 1)]u(2) + [\delta(3 - 3) + 2\delta(3 - 3 - 1)]u(3) = 2\delta(0)u(2) = 2 \times 1 \times 3 = 6 \]
The output will be zero before $k = 1$ as the input is only applied at time $k = 1$ and, as the system is causal and has no initial conditions, no output will be observed before this time.

The output will be zero after $k = 3$ because the input is zero after $k = 2$ and the system’s impulse response only has a one “sample” memory.

2. The solution to this question is similar to that given above and involves a direct application of the convolution formula given in the question. Observe that, similar to before:

\[ y_1(k) = \sum_{n=-\infty}^{\infty} [\delta(k-n) + 2\delta(k-n-1)]u_1(n) \]

Evaluating this, as before for $k = 1, 2, 3$ and with $u_1(k)$ we have

\[
y_1(1) = \sum_{n=-\infty}^{\infty} [\delta(1-n) + 2\delta(1-n-1)]u(n) \\
= [\delta(1-0) + 2\delta(1-1)]u(0) \\
+ [\delta(1-1) + 2\delta(1-1-1)]u(1) \\
= \delta(0)u(1) = 1 \times 2 = 2
\]

\[
y_1(2) = \sum_{n=-\infty}^{\infty} [\delta(2-n) + 2\delta(2-n-1)]u(n) \\
= [\delta(2-0) + 2\delta(2-1)]u(0) \\
+ [\delta(2-1) + 2\delta(2-1-1)]u(1) \\
+ [\delta(2-2) + 2\delta(2-2-1)]u(2) \\
= 2\delta(0)u(1) + \delta(0)u(2) = 2 \times 1 \times 2 \\
= 4
\]

\[
y_1(3) = \sum_{n=-\infty}^{\infty} [\delta(3-n) + 2\delta(3-n-1)]u(n) \\
= [\delta(3-0) + 2\delta(3-1)]u(0) \\
+ [\delta(3-1) + 2\delta(3-1-1)]u(1) \\
+ [\delta(3-2) + 2\delta(3-2-1)]u(2) \\
+ [\delta(3-3) + 2\delta(3-3-1)]u(3) \\
= 0
\]

Carrying out a similar calculation as this for $y_2(k) = G[u_2(k)]$, yields

\[
y_2(1) = 0, \quad y_2(2) = 3, \quad y_2(3) = 6
\]

Thus it can easily be seen that $y_1 + y_2 = y = G[u_1 + u_2]$ as expected.
3. We need to prove that \( h(t) \ast u(t) = u(t) \ast h(t) \)

\[
u(t) \ast h(t) = \int_{-\infty}^{\infty} u(t - \tau)h(\tau)d\tau
\]

Now let \( \lambda = t - \tau \), which implies that

\[
\tau = t - \lambda \quad \text{and} \quad d\tau = -d\lambda
\]

So we have

\[
u(t) \ast h(t) = -\int_{-\infty}^{\infty} h(t - \lambda)u(\lambda)d\lambda
\]

\[
= -\int_{t-\lambda=-\infty}^{t-\lambda=\infty} h(t - \lambda)u(\lambda)d\lambda
\]

\[
= -\int_{-\lambda=-\infty}^{\lambda=\infty} h(t - \lambda)u(\lambda)d\lambda
\]

\[
= -\int_{-\infty}^{\infty} h(t - \lambda)u(\lambda)d\lambda
\]

\[
= -\left( \int_{-\infty}^{\infty} h(t - \lambda)u(\lambda)d\lambda \right)
\]

\[
= \int_{-\infty}^{\infty} h(t - \lambda)u(\lambda)d\lambda
\]

\[
= h(t) \ast u(t)
\]

Key point: there is no significance in the dummy \textit{variable} used for integration: \( \tau, \lambda, x \ldots \), it makes no difference.

4. We need to prove that \( x(t) \ast [y(t) \ast z(t)] = [x(t) \ast y(t)] \ast z(t) \). First let \( w(t) = y(t) \ast z(t) \). Then we have

\[
x(t) \ast [y(t) \ast z(t)] = x(t) \ast w(t)
\]

\[
= \int_{-\infty}^{\infty} w(t - \tau)x(\tau)d\tau
\]

Now it follows that

\[
w(t - \tau) = y(t - \tau) \ast z(t - \tau)
\]

\[
= \int_{-\infty}^{\infty} z(t - \tau - \lambda)y(\lambda)d\lambda
\]

and thus that we have:

\[
x(t) \ast [y(t) \ast z(t)] = \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} z(t - \tau - \lambda)y(\lambda)d\lambda d\tau
\]

\[
= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} x(\tau)y(\lambda)z(t - \tau - \lambda)d\lambda \right) d\tau
\]

Next consider: \([x(t) \ast y(t)] \ast z(t)\) and let \( v(t) = x(t) \ast y(t) \). So we have

\[
v(t) \ast z(t) = \int_{-\infty}^{\infty} v(\tau)z(t - \tau)d\tau
\]
But

\[ v(t) = x(t) \ast y(t) = \int_{-\infty}^{\infty} x(\lambda)y(t - \lambda)d\lambda \]

and hence

\[ v(\tau) = \int_{-\infty}^{\infty} x(\lambda)y(\tau - \lambda)d\lambda \]

and hence

\[ v(t) \ast z(t) = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} x(\lambda)y(\tau - \lambda)d\tau \right) z(t - \tau)d\tau \]

Now change variables:

\[ \sigma = t - \tau \Rightarrow \begin{cases} \tau = t - \sigma \\ d\tau = -d\sigma \end{cases} \]

Then we have

\[ v(t) \ast z(t) = -\int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} x(\lambda)y(t - \sigma - \lambda)d\lambda \right) z(\sigma)d\sigma \]

\[ = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} x(\lambda)y(t - \sigma - \lambda)z(\sigma)d\lambda \right) d\sigma \]

Finally, change variables again:

\[ \alpha = t - \sigma - \lambda \Rightarrow \begin{cases} \sigma = t - \lambda - \alpha \\ d\sigma = -d\alpha \end{cases} \]

gives

\[ v(t) \ast z(t) = -\int_{+\infty}^{-\infty} \left( \int_{-\infty}^{\infty} x(\lambda)y(\alpha)z(t - \lambda - \alpha)d\lambda \right) d\alpha \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\lambda)y(\alpha)z(t - \lambda - \alpha)d\lambda d\alpha \]

Notice that the above equation is identical to (2) except with different (and not significant variables of integration).