Aircraft Stability and Performance:
Longitudinal Static Stability - supporting maths

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Longitudinal static stability

Basic condition for longitudinal static stability

\[
\frac{dM}{d\alpha} < 0 \quad \left( \frac{dC_M}{d\alpha} < 0 \right)
\]

Diagram:

- Nose up
- Nose down

“trim” angle of attack

Statically stable aircraft

Statically unstable aircraft
Longitudinal static stability - alternative expressions

- Non-dimensionalise stability condition

\[ \frac{dM}{d\alpha} = \frac{1}{2} \rho S \bar{c} V^2 \frac{dC_M}{d\alpha} > 0 \]

- Equivalent condition for stability is

\[ \frac{dC_M}{d\alpha} < 0 \]

- But...

\[ \frac{dC_M}{d\alpha} = \frac{dC_M}{dC_L} \frac{dC_L}{d\alpha} > 0 \]

Equivalent condition for static stability

\[ \frac{dC_M}{dC_L} < 0 \]

Pitching moment:

\[ M = \frac{1}{2} \rho S \bar{c} C_M V^2 \]

Pitching moment coefficient

\[ C_M \]

(Mean chord length \( \bar{c} \))
Aerodynamic Centre - simplified introduction

When a (2D) object travels through a fluid, the forces and moments acting on that object can be accounted for by a net aerodynamic force and an accompanying aerodynamic moment applied at a single point.

- Aerodynamic moment varies, depending on the point where the force is chosen to act.
- For airfoils...obvious choice is the centre of pressure (c.p.): point where $M = 0$ (i.e. $C_M = 0$).
- ...but $M(\alpha)$ varies as a function of $\alpha$ i.e. when $\alpha$ changes, $M$ changes. $\Rightarrow$ location of centre of pressure changes when $\alpha$ changes.
Aerodynamic centre - simplified introduction

- Aerodynamic centre (a.c.): similar idea to c.p. but crucially $M$ is not a function of $\alpha$

$$\frac{dC_M}{d\alpha} = 0 \quad \text{or} \quad \frac{dC_M}{dC_L} = 0$$

($M =$ pitching moment, $C_M =$ pitching moment coefficient)

⇒ a.c. remains in same location regardless of angle of attack
⇒ Greatly simplifies analysis

Most aircraft: aerodynamic centre (a.c.) $\approx 0.25\bar{c}$ from leading edge of wing.
Pitching moment - illustration

\[ \overline{c} \] mean chord length

\[ h_0 \overline{c} \] distance from leading edge to a.c.

\[ h \overline{c} \] distance from leading edge to c.g.

\[ l_t \] distance from a.c. to tail

\[ L_w (L_t) \] Lift due to wings (tail)

\[ M_w (M_t) \] Pitching moment due to wings (tail)

Assumptions

- \( T, D \) pass through c.g., a.c. and tail a.c. → neglect moments due to \( T \) and \( D \).

- \( \alpha \) is small → trig. terms simplify substantially
Pitching moment - analysis without tailplane

- Assume $L_t = M_t = 0$ and take moments around a.c

\[ M = M_w + (h - h_0)\bar{c}mg \]
\[ = M_w + (h - h_0)\bar{c}L_w \]

- Normalise with respect to $1/2\rho SV^2\bar{c}$

\[ C_M = C_{Mw} + (h - h_0)C_{Lw} \]

- Examine derivative

\[ \frac{dC_M}{dC_{Lw}} = (h - h_0) > 0 \quad \text{(typically)} \]

- Aircraft is not statically stable!

We have discovered the reason why most aircraft typically have tail planes!
Adding the tail plane

Triangle BCD: $\theta + \epsilon + \alpha_t = 180$
Triangle ACE: $\theta + \alpha + \eta_t = 180$
Hence: $\alpha_t = \alpha + \eta_t - \epsilon$

When measured w.r.t. n.l.l:

\begin{align*}
C_{Lw} &= C_{Lw\alpha} \alpha \\
C_{Lt} &= C_{Lt\alpha_t} \alpha_t
\end{align*}

<table>
<thead>
<tr>
<th>$\alpha_t$</th>
<th>tail plane incidence</th>
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<tbody>
<tr>
<td>$\eta_t$</td>
<td>tail plane setting angle</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>downwash angle</td>
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Adding the tail plane

- Basic idea: express tail plane lift coefficient ($C_{Lt}$) as a function of wing lift coefficient, $C_{Lw}$
- Note: downwash angle expressed as a function of wing incidence:

$$\epsilon = \alpha \frac{d\epsilon}{d\alpha}$$

- Hence

$$\alpha_t = \left(1 - \frac{d\epsilon}{d\alpha}\right) \alpha + \eta_t$$

- Recalling that $C_{Lw} = C_{Lw\alpha} \alpha$:

$$\alpha_t = \frac{C_{Lw}}{C_{Lw\alpha}} \left(1 - \frac{d\epsilon}{d\alpha}\right) + \eta_t$$

- Now recalling $C_{Lt} = C_{Lt\alpha_t} \alpha_t$:

$$C_{Lt} = \frac{C_{Lt\alpha_t}}{C_{Lw\alpha}} \left(1 - \frac{d\epsilon}{d\alpha}\right) C_{Lw} + C_{Lt\alpha_t} \eta_t$$
Pitching moment - analysis including tailplane

- Take moments around wing a.c

\[ M = M_w + (h - h_0)\bar{c}L_w + M_t - l_tL_t \]

- Normalise with respect to \( \frac{1}{2}\rho SV^2\bar{c} \)

\[
C_M = C_{Mw} + (h - h_0)C_{Lw} + \frac{M_t}{\frac{1}{2}\rho S\bar{c}V^2} - \frac{1}{2}\rho l_tS_tV^2C_{Lt} \\
\quad = C_{Mw} + (h - h_0)C_{Lw} + C_{Mt} - \frac{S_t l_t}{S\bar{c}} C_{Lt} \\
\quad = C_{Mw} + (h - h_0)C_{Lw} + C_{Mt} - \frac{S_t l_t}{S\bar{c}} \left( \frac{C_{Lt\alpha_t}}{C_{Lw\alpha}} \left( 1 - \frac{d\epsilon}{d\alpha} \right) C_{Lw} + C_{L\alpha_t}\eta_t \right)
\]

- Define tail plane volume coefficient \( \bar{V}_t = l_tS_t/S\bar{c} \) and differentiate w.r.t. \( C_{Lw} \)

\[
\frac{dC_M}{dC_{Lw}} = (h - h_0) - \bar{V}_t \frac{C_{Lt\alpha_t}}{C_{Lw\alpha}} \left( 1 - \frac{d\epsilon}{d\alpha} \right)
\]

- Thus: \( 1 - \frac{d\epsilon}{d\alpha} > 0 \) and \( \bar{V}_t \frac{C_{Lt\alpha_t}}{C_{L\alpha}} \) sufficiently large \( \Rightarrow \) static stability
Static margins

Neutral point \((h_n)\): position which, if c.g. were placed here, would give neutral static stability

- For neutral static stability: \(\frac{dC_M}{dC_L} \approx \frac{dC_M}{dC_{Lw}} = 0\). Hence neutral point is:

\[
h_n := h_0 + \bar{V}_t \frac{C_{Lt\alpha_t}}{C_{Lw\alpha}} \left( 1 - \frac{d\epsilon}{d\alpha} \right)
\]

- Using definition of neutral point, condition for static stability can be expressed:

\[
\frac{dC_M}{dC_L} = h - h_n < 0 \iff H_n := h_n - h > 0
\]

- \(H_n\) is the *static stability margin*: \(h_n > h\) for stability
Static margins

For static stability: c.g. must lie forward of neutral point
Main reason for tail plane is not lift - it is static stability