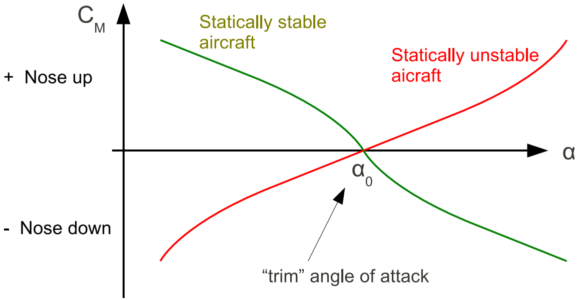
Aircraft Stability and Performance: Longitudinal Static Stability - supporting maths

Dr. M. Turner

Longitudinal static stability

Basic condition for longitudinal static stability

$$\frac{dM}{d\alpha} < 0 \quad \left(\frac{dC_M}{d\alpha} < 0 \right)$$



Longitudinal static stability - alternative expressions

- Non-dimensionalise stability condition

$$\frac{dM}{d\alpha} = \underbrace{\frac{1}{2}\rho S\bar{c}V^2}_{>0}\frac{dC_M}{d\alpha}$$

- Equivalent condition for stability is

$$\frac{dC_M}{d\alpha} < 0$$

- But...

$$\frac{dC_M}{d\alpha} = \frac{dC_M}{dC_L}\underbrace{\frac{dC_L}{d\alpha}}_{>0}$$

Pitching moment:

$$M = \frac{1}{2}\rho S\bar{c}C_M V^2$$

Pitching moment coefficient

$$C_M$$

(Mean chord length $\bar{c}$)

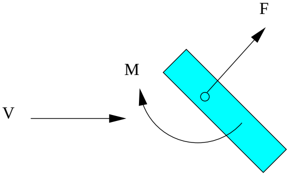
Equivalent condition for static stability

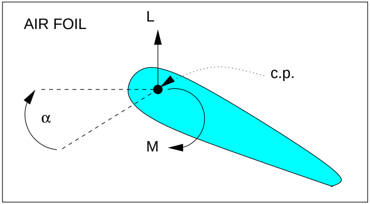
$$\frac{dC_M}{dC_L} < 0$$

Aerodynamic Centre - simplified introduction

> When a (2D) object travels through a fluid, the forces and moments acting on that object can be accounted for by a net aerodynamic force and an accompanying aerodynamic moment applied at a single point

- Aerodynamic moment varies, depending on the point where the force is chosen to act.
- For air foils...obvious choice is the centre of pressure (c.p.): point where $M = 0$ (i.e. $C_M = 0$)
- ...but $M(\alpha)$ varies as a function of α i.e. when α changes, M changes. $\Rightarrow$ location of centre of pressure changes when α changes.





Aerodynamic centre - simplified introduction

- Aerodynamic centre (a.c.): similar idea to c.p. but *crucially* M is not a function of α

$$\frac{dC_M}{d\alpha} = 0 \quad \text{or} \quad \frac{dC_M}{dC_L} = 0$$

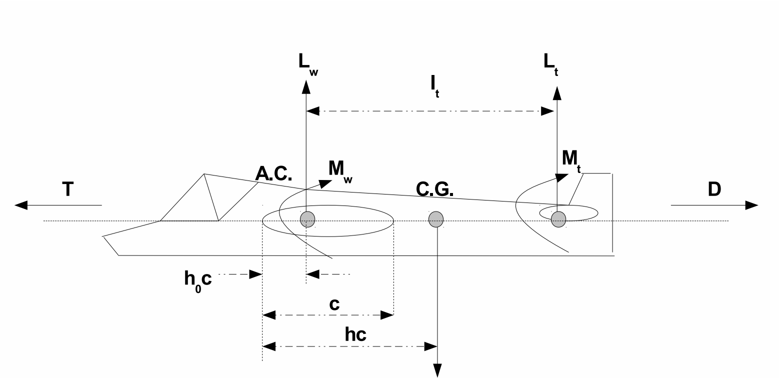
(M= pitching moment, C_M= pitching moment coefficient)

$\Rightarrow$ a.c. remains in same location regardless of angle of attack

$\Rightarrow$ Greatly simplifies analysis

Most aircraft: aerodynamic centre (a.c.) $\approx 0.25\bar{c}$ from leading edge of wing.

Pitching moment - illustration



$\bar{c}$	mean chord length
$h_0\bar{c}$	distance from leading edge to a.c.
$h\bar{c}$	distance from leading edge to c.g.
l_t	distance from a.c. to tail
L_w (L_t)	Lift due to wings (tail)
M_w (M_t)	Pitching moment due to wings (tail)

Assumptions

- T, D pass through c.g., a.c. and tail a.c
  $\rightarrow$ neglect moments due to T and D.
- α is small
  $\rightarrow$ trig. terms simplify substantially

Pitching moment - analysis without tailplane

- Assume $L_t = M_t = 0$ and take moments around a.c

$$\begin{aligned} M &= M_w + (h - h_0)\bar{c}mg \\ &= M_w + (h - h_0)\bar{c}L_w \end{aligned}$$

- Normalise with respect to $1/2\rho SV^2\bar{c}$

$$C_M = C_{Mw} + (h - h_0)C_{Lw}$$

- Examine derivative

$$\frac{dC_M}{dC_{Lw}} = (h - h_0) > 0 \quad \text{(typically)}$$

- Aircraft is not statically stable!

Lift due to wing:

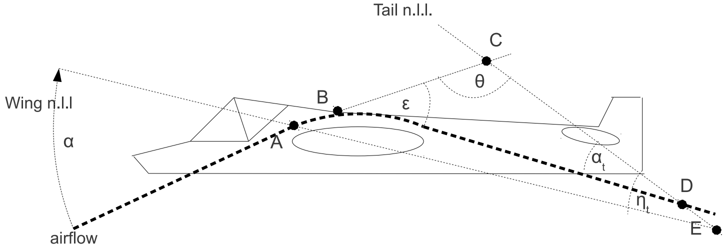
$$L_w = \frac{1}{2}C_{Lw}\rho SV^2$$

Pitching moment due to wing:

$$M_w = \frac{1}{2}C_{Mw}\rho SV^2\bar{c}$$

We have discovered the reason why most aircraft typically have tail planes!

Adding the tail plane



When measured w.r.t. n.l.l:

- Triangle BCD: $\theta + \epsilon + \alpha_t = 180$
- Triangle ACE: $\theta + \alpha + \eta_t = 180$
- Hence: $\alpha_t = \alpha + \eta_t - \epsilon$

$$C_{Lw} = C_{Lw\alpha}\alpha$$
$$C_{Lt} = C_{Lt\alpha_t}\alpha_t$$

α_t	tail plane incidence
η_t	tail plane setting angle
ϵ	downwash angle

Adding the tail plane

- Basic idea: express tail plane lift coefficient (C_{Lt}) as a function of wing lift coefficient, C_{Lw}
- Note: downwash angle expressed as a function of wing incidence:

$$\epsilon = \alpha \frac{d\epsilon}{d\alpha}$$

- Hence

$$\alpha_t = \left(1 - \frac{d\epsilon}{d\alpha}\right)\alpha + \eta_t$$

- Recalling that $C_{Lw} = C_{Lw\alpha}\alpha$:

$$\alpha_t = \frac{C_{Lw}}{C_{Lw\alpha}}\left(1 - \frac{d\epsilon}{d\alpha}\right) + \eta_t$$

- Now recalling $C_{Lt} = C_{Lt\alpha_t}\alpha_t$:

$$C_{Lt} = \frac{C_{Lt\alpha_t}}{C_{Lw\alpha}}\left(1 - \frac{d\epsilon}{d\alpha}\right)C_{Lw} + C_{Lt\alpha_t}\eta_t$$

Pitching moment - analysis including tailplane

- Take moments around wing a.c

$$M = M_w + (h - h_0)\bar{c}L_w + M_t - l_t L_t$$

- Normalise with respect to $1/2\rho S V^2 \bar{c}$

$$\begin{aligned} C_M =& C_{Mw} + (h - h_0)C_{Lw} + \frac{M_t}{1/2\rho S\bar{c}V^2} - \frac{1/2\rho l_t S_t V^2 C_{Lt}}{1/2\rho S V^2 \bar{c}} \\ =& C_{Mw} + (h - h_0)C_{Lw} + C_{Mt} - \frac{S_t l_t}{S\bar{c}} C_{Lt} \\ =& C_{Mw} + (h - h_0)C_{Lw} + C_{Mt} - \frac{S_t l_t}{S\bar{c}} \left(\frac{C_{Lt\alpha_t}}{C_{Lw\alpha}} \left(1 - \frac{d\epsilon}{d\alpha}\right) C_{Lw} + C_{L\alpha_t}\eta_t \right) \end{aligned}$$

- Define *tail plane volume coefficient* $\bar{V}_t = l_t S_t / S\bar{c}$ and differentiate w.r.t. C_{Lw}

$$\frac{dC_M}{dC_{Lw}} = (h - h_0) - \bar{V}_t \frac{C_{Lt\alpha_t}}{C_{Lw\alpha}} \left(1 - \frac{d\epsilon}{d\alpha}\right)$$

- Thus: $1 - \frac{d\epsilon}{d\alpha} > 0$ and $\bar{V}_t \frac{C_{L\alpha_t}}{C_{L\alpha}}$ sufficiently large $\Rightarrow$ static stability

Static margins

Neutral point (h_n): position which, if c.g. were placed here, would give neutral static stability

- For neutral static stability: $\frac{dC_M}{dC_L} \approx \frac{dC_M}{dC_{Lw}} = 0$. Hence neutral point is:

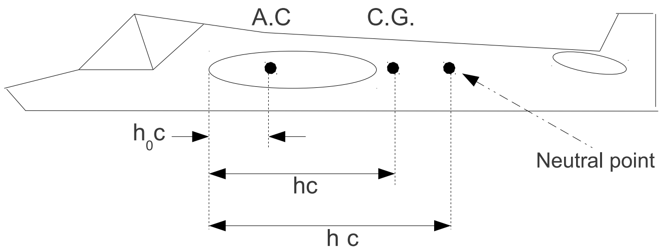
$$h_n := h_0 + \bar{V}_t \frac{C_{Lt\alpha_t}}{C_{Lw\alpha}} \left(1 - \frac{d\epsilon}{d\alpha}\right)$$

- Using definition of neutral point, condition for static stability can be expressed:

$$\frac{dC_M}{dC_L} = h - h_n < 0 \quad \Leftrightarrow \quad H_n := h_n - h > 0$$

- H_n is the *static stability margin*: $h_n > h$ for stability

Static margins



- For static stability: c.g. must lie forward of neutral point
- Main reason for tail plane is not lift - it is static stability