Accounting for uncertainty in anti-windup synthesis

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Abstract

This paper describes an approach to synthesising anti-windup compensators which can improve the behaviour of systems subject to actuator saturation while also taking into account uncertainty in the system. The class of uncertainty considered is reasonably large and, moreover, is of the type often used in practice and often considered in linear robust control. The development of the ideas makes use of a ‘decoupled’ representation of an anti-windup scheme which is useful for comparing the results from standard approaches to anti-windup compensation to those compensators obtained using the new robust approach. An interesting, but perhaps not surprising feature of these results is that the often-criticised internal model control (IMC) anti-windup solution emerges as an ‘optimally’ robust solution.

1 Introduction

The problems associated with robustness, or lack thereof, to model uncertainty and the problems associated with actuator saturation have focused the minds of control engineers for over a decade now. Remarkably though, these problems have often been considered in isolation and the authors are only aware of one substantial body of work which attempts to unify some of the results ([11]). On a second look though, perhaps this is not so odd: actuator saturation could be considered, at a crude level, as model uncertainty and taken into account in the same way as other uncertainty; this could be handled quite routinely in the $\mathcal{H}_\infty$ and $\mu$-synthesis approaches to controller design. The problem with this approach is that the introduction of another uncertainty into the optimisation problems has the tendency to make the resulting design overly conservative and therefore potentially of low performance.

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In contrast those engineers studying the behaviour of systems subject to actuator saturation have, by and large, chosen to ignore the effect of model uncertainty in their proposals. In anti-windup particularly this has been the case, with the prevailing attitude being to assume that if the nominal linear design is robust, then the anti-windup compensated system will inherit this robustness. This makes some intuitive sense, although it seems more logical to hypothesise that nominal linear robustness is a necessary, but not sufficient, condition for the robustness of the overall anti-windup compensated (nonlinear) system.

The origin of this work has two sources. The first inspiration came from the work of [11] which contains a useful account of the pioneering early work on the subject of linear systems subject to actuator saturation and model uncertainty. Contained within this book are a collection of papers which address the analysis and synthesis of controllers which, a priori, account for actuator saturation and also ensure some degree of robustness for the feedback interconnection. Most of these papers deal with the same type of uncertainty: parametric or state-space uncertainty, generally of the form

\[ \dot{x} = Ax + B \text{sat}(u) + \Delta(x,t) \] (1)

Although this type of parametric uncertainty is certainly useful, in practice it is quite limited in scope and is not very useful for capturing unmodelled dynamics which can be more of an obstacle than their modelled, but uncertain, counterparts.

The second point of origin was the paper [14] where some initial ideas were put forward to anti-windup compensation in the presence of model uncertainty. This current paper is a continuation of those ideas, but with more constructive techniques for synthesising anti-windup compensators. It is important to remark that the motivation for this work, and also for that of [14] is practical: in our experience in the aerospace and hard-disk control fields, it has really been the unmodelled dynamics which have caused the most difficulty in appropriate controller synthesis and while present, the parametric uncertainty, has played a less prominent role (it is often easily countered by large enough low frequency gain).

The aims of this paper are two-fold. Firstly it aims to bring robustness to the fore in anti-windup compensation, where, except for [12], it has had little prominence. Secondly, this paper aims to promote the use of a type of uncertainty which is closer to that often used in practice (and close to that used in linear robust control theory, which has shown itself to be successful in recent years).

Notation used in the paper is standard. In particular we define the induced \( L_2 \) norm, or finite \( L_2 \) gain, of an operator \( \mathcal{H} \) as \( \| \mathcal{H} \|_{i,2} := \sup_{\| x \| \in \mathbb{L}_2} \left\| \mathcal{H} x \right\|_2 \) where \( \| x \|_2 = \sqrt{\int_0^\infty \| x(t) \|^2 dt} \) is the \( L_2 \) norm of the vector \( x(t) \) and \( \| x \| \) is its Euclidean norm. The \( \mathcal{H}_\infty \) norm for a linear operator \( P \) is defined as \( \| P \|_\infty := \sup_{\omega} \sigma(P(j\omega)) \) where \( \sigma(\cdot) \) denotes the maximum singular value and \( P(j\omega) \) is the frequency response matrix associated with the linear operator \( P \). We do not explicitly distinguish between a linear operator and its transfer function. Equivalently, the \( \mathcal{H}_\infty \) norm may be defined as \( \| P \|_\infty = \| P \|_{i,2} \).
2 A general anti-windup framework

We take as our starting point the scheme introduced in [16], where anti-windup is interpreted as choosing an appropriate transfer function matrix $M(s)$. The scheme is shown in Figure 1 where $G(s) = [G_1(s) \, G_2(s)]$ is the plant and $K(s) = [K_1(s) \, K_2(s)]$ is the controller. This can be re-drawn as the decoupled scheme in Figure 2. Here we have used the fact that the saturation function and the deadzone functions are related by the identity

$$\text{sat}(u) = u - Dz(u)$$

(2)

where

$$\text{sat}(u) = \begin{bmatrix} \text{sat}_1(u_1) \\ \vdots \\ \text{sat}_m(u_m) \end{bmatrix} \quad \text{Dz}(u) = \begin{bmatrix} Dz_1(u_1) \\ \vdots \\ Dz_m(u_m) \end{bmatrix}$$

(3)

and $\text{sat}_i(u_i) = \text{sign}(u_i) \min(|u_i|, \bar{u}_i) \quad \forall i$ and $Dz_i(u_i) = \text{sign}(u_i) \max(0, |u_i| - \bar{u}_i)\forall i$. Also $\bar{u}_i > 0 \quad \forall i \in \{1, \ldots, m\}$.

In [17], it was shown that most anti-windup schemes can be interpreted as certain choices of $M(s)$ and therefore schemes such as the Hanus conditioning scheme ([4]), the high gain approach ([1, 7]) and such like can be analysed in terms of Figure 2. The advantages of viewing anti-windup in terms of Figure 2 is that the nominal linear performance is separated from the nonlinear part of the scheme and moreover, the stability of the scheme is dependent on the stability of the nonlinear loop. From Figure 2, it can be seen that the performance of the anti-windup compensator is intimately related to the mapping $\mathcal{T}_p : u_{\text{lin}} \mapsto y_d$: if the norm of this mapping is small, then the anti-windup compensator is successful at keeping performance close to linear (which we assume is the desired performance). In [15], the $L_2$ gain of $\mathcal{T}_p$ was minimised using a system of linear matrix inequalities and, furthermore, $M(s)$ was chosen such that it corresponded to static or low order anti-windup compensators (The discrete and sampled-data versions of this problem can be found in [13] and [5]). It was demonstrated, using suitable examples, in [15] that direct minimisation of $\mathcal{T}_p$ was central to good anti-windup performance and compensators designed according to the ideas in [15] seemed to perform a least as well, and often better, than most other anti-windup compensators.

2.1 With uncertainty

Let us now consider the configuration in Figure 3 where $\hat{G}$ is the true plant given by $\hat{G}(s) = [G_1(s) \, G_2(s) + \Delta_G(s)]$, where $G(s) = [G_1(s) \, G_2(s)]$ is the model of the plant with which we work and $\Delta_G$ is additive uncertainty to the feedback part\(^1\) which we assume is stable and linear. Other types of uncertainty such as output-
When uncertainty is present in the system, the appealing decoupled structure of the original scheme is lost. Figure 4 shows an equivalent representation of Figure 3. Note the term $\Delta_G M : \tilde{u} \mapsto y_\Delta$ destroys the decoupling of the linear system and nonlinear loop.
2.2 Assumptions

In this section we list and explain the assumptions on which the remainder of the paper is based.

1. The open-loop plant, $G(s)$, is asymptotically stable. The reason for this assumption is that this work seeks to establish global results and it is well known that, for constrained input systems, global asymptotic stability with finite-gain stability can only be established providing the system is open-loop asymptotically stable.

2. The (linear) uncertainty $\Delta G(s)$ is asymptotically stable. This mirrors the case in standard $\mathcal{H}_\infty$ control theory where the perturbations are assumed stable. This greatly simplifies the work and is necessary for the small-gain approach we take.

3. The nominal linear closed-loop system is robustly asymptotically stable. By this, we mean that, when the saturation nonlinearity is replaced by the identity operator, the closed-loop system is stable and furthermore can tolerate a certain amount of uncertainty ($\|\Delta G\|_\infty < \gamma$, where $\gamma = \|(I - K_2G_2)^{-1}K_2\|_\infty$) before becoming unstable. This essentially amounts to assuming that the design of the linear controller $K(s)$ is “good” in the sense that it robustly stabilises the system. We also assume that the nominal linear closed-loop is well-posed.

4. The nominal linear closed-loop yields desirable performance. As is common in the anti-windup literature, it is assumed that the linear closed-loop yields desired performance and the performance of the anti-windup compensator can be measured against the deterioration of this performance when the control signals saturate. This is related to the foregoing point in the sense that we also assume that the linear closed-loop yields desirable robustness properties and therefore the performance of the anti-windup compensator can
also be assessed against its preservation of the linear system’s robustness properties.

On the basis of these assumptions three features are evident from Figure 4:

1. If $\Delta G$ is small in some sense, then the robustness of the anti-windup scheme is similar to that of the nominal, unconstrained linear system (via a Small Gain argument).

2. If the mapping from $u_{lin} \mapsto M\hat{u}$ is small, again, the robustness of the anti-windup system is similar to that of the nominal linear system. (again using a Small gain argument). So in other words the map $u_{lin} \mapsto M\hat{u}$ contains important robustness information.

3. The robustness of the system with anti-windup compensation can never be better than the robustness of the linear system. Denoting the ‘modified’ uncertainty $\hat{\Delta}_G : u_{lin} \mapsto y_\Delta$, this follows by noting that $\|\Delta_G\|_{\infty} = \|\hat{\Delta}_G\|_{l,2} \leq \|\hat{\Delta}_G\|_{l,2}$ (by using a contradiction argument). So, in a sense, the retention of the linear system’s robustness can be considered as an optimal property. This will be discussed in more detail later.

### 3 Special Case: IMC anti-windup

Before we explore the consequences of uncertainty in anti-windup further, it is interesting to consider a special case: the much-maligned IMC anti-windup scheme. IMC anti-windup was introduced in [19] as an anti-windup methodology but many examples in the literature have shown it to be a very poorly performing anti-windup scheme (see eg [2] and others). This can be easily seen by viewing IMC anti-windup in Figure 2: to obtain IMC-anti-windup we simply choose $M = I$. The nonlinear ‘loop’ becomes simply the deadzone operator, and the disturbance filter becomes the open-loop plant. Hence the IMC performance will be poor if the open-loop plant has lightly damped or nonminimum phase zeros.

As is often the case in linear robust control theory, there is a trade-off between performance and robustness and this seems to extend to anti-windup compensation. Consider uncertain anti-windup in Figure 4 and choose $M = I$, then again the nonlinear loop degenerates to the deadzone function and the troublesome term, which destroys the de-coupling of the linear and nonlinear parts of the system, simply becomes the uncertainty, $\Delta G$. For convenience, this scenario is re-drawn in Figure 5. Note that for consistency we have retained the notation $y_{lin}$ and $u_{lin}$, although it should be understood that these signals are no longer generated by a purely linear system.

Now, nominally, assuming no saturation, simple small gain analysis shows that we have stability robustness against all input additive uncertainty

\[ \|\Delta_G\|_{\infty} < \frac{1}{\gamma} \] (4)
where \( \| (I - K_2 G_2) K_2 \|_{\infty} := \gamma \). Carrying out a small gain analysis on Figure 5 we see we have stability providing that the ‘modified’ nonlinear uncertainty \( \tilde{\Delta}_G \) satisfies

\[
\| \tilde{\Delta}_G \|_{i,2} < \frac{1}{\gamma} \quad (5)
\]

But as

\[
\| \tilde{\Delta}_G \|_{i,2} \leq \| \Delta_G \|_{\infty}\|I - D\tilde{z}(\cdot)\|_{i,2} = \| \Delta_G \|_{\infty}\|\text{sat}(\cdot)\|_{i,2}
= \| \Delta_G \|_{\infty} \quad (6)
\]

we see that we have stability robustness to all uncertainty satisfying inequality (4). However, as we also have \( \| \Delta_G \|_{\infty} \leq \| \tilde{\Delta}_G \|_{i,2} \) we must have that \( \| \tilde{\Delta}_G \|_{i,2} = \| \Delta_G \|_{\infty} \). In other words, the IMC anti-windup scheme is guaranteed to be robustly stable for the same class of additive uncertainties as the nominal linear system. Recall, that it is not possible for an anti-windup scheme to be more robust than the nominal linear system because much of the anti-windup scheme’s time is spent operating as a linear system. So, in a sense, the retention of the linear system’s robustness properties is optimal. Hence, although IMC schemes can be criticised for their performance, they are in fact optimally robust as far as stability is concerned!

4 General Case

4.1 A stability robustness criterion

From Figure 4, we have that

\[
y_{lin} = G_1 d + G_2 u_{lin} + \Delta_G [u_{lin} - M\tilde{F}(u_{lin})] = G_1 d + G_2 u_{lin} + \tilde{\Delta}_G(u_{lin}) \quad (8)
\]
where $\mathcal{F}(u_{\text{lin}})$ is the map from $u_{\text{lin}}$ to $\tilde{u}$. Carrying out a small gain analysis we see that the system is robust against all additive perturbations such that

$$\|\tilde{\Delta}_G\|_{2} = \|\Delta_G[I - M\mathcal{F}(u_{\text{lin}})]\|_{2} < \frac{1}{\gamma}$$  \hspace{1cm} (9)$$

So nominal robustness is retained if $\|[I - M\mathcal{F}(u_{\text{lin}})]\|_{2} \leq 1$. However as $\mathcal{F}(u_{\text{lin}}) = 0$ around $u_{\text{lin}} = 0$ (as it contains the deadzone), $\|[I - M\mathcal{F}(u_{\text{lin}})]\|_{2}$ can never be strictly less than unity. Again this conclusion coincides with our prior discussion as we could not expect an anti-windup scheme to yield greater robustness margins than the linear system upon which it is constructed. This also serves as justification for the IMC scheme, although it is unlikely to be the unique compensator which achieves this optimality.

4.2 Stability robustness optimisation

![Diagram](image)

Figure 6: Robustness optimisation for general anti-windup schemes: graphical representation of $\mathcal{T}_r$

In this section, we shall consider robustness optimisation using full-order anti-windup compensators. As suggested in [17] we choose $M(s)$ to be part of a right-coprime factorisation of $G_2(s) = N(s)M^{-1}(s)$ (this in fact is a dual result to that of [9], where anti-windup is described as a left coprime factorisation of the controller) and attempt to choose a particular factorisation such that robustness is optimised.

In order to preserve as much robustness as possible we would like to minimise $\|\mathcal{T}_r\|_{2} := \|[I - M\mathcal{F}(u_{\text{lin}})]\|_{2}$. Diagramatically this can be shown as Figure 6 where we want to minimise the $\mathcal{L}_2$ gain from $u_{\text{lin}}$ to $z$. As mentioned in [15], this optimisation is typically a difficult optimisation problem to solve, so instead we shall be content to ensure a certain $\mathcal{L}_2$ gain bound holds for the map $\mathcal{T}_r$.

Given a plant realisation

$$G_2 \approx \begin{bmatrix} A_p & B_p \\ C_p & D_p \end{bmatrix}$$ \hspace{1cm} (10)$$

all full-order right coprime factorisations can be described as
where $F$ is chosen such that $A_p + B_p F$ is a Hurwitz matrix.

To ensure robustness, that is to ensure that $\| \mathcal{T}_r \|_{l,2} < \gamma$, it suffices for the following inequality to hold for sufficiently small $\gamma$:

$$J = \frac{d}{dt} x^T P x + \| z \|^2 - \gamma^2 \| u_{lin} \|^2 < 0$$

(12)

where $x$ is the state vector associated with the realisation of $[M - I, M]$. As shown in, for example [3, 15] this ensures that the $L_2$ gain from $u_{lin}$ to $z$ is less than $\gamma$ and that the system in Figure 6 is asymptotically stable.

As the deadzone nonlinearity belongs to the Sector $[0, I]$ (see [8], chapter 10), we make use of the inequality

$$2 \tilde{u} W (u_{lin} - u_d - \tilde{u}) \geq 0$$

(13)

to form the modified cost function

$$J = \frac{d}{dt} x^T P x + \| z \|^2 - \gamma^2 \| u_{lin} \|^2 + 2 \tilde{u} W (u_{lin} - u_d - \tilde{u})$$

(14)

If $J < 0$, this implies that $J < 0$. Evaluating $J$ in a similar manner to [15] (see also [13, 5] for the discrete and sampled data cases) yields the following LMI

$$\begin{bmatrix}
QA' + A_p Q + L' B_p' + B_p L & B_p U - L' & 0 & L' \\
* & -2U & I & U \\
* & * & -\mu I & -I \\
* & * & * & -I
\end{bmatrix} < 0$$

(15)

in the variables $Q > 0, U = \text{diag}(\nu_1, \ldots, \nu_m) > 0, L, \mu > 0$

Satisfaction of this LMI means that inequality (14) is satisfied and hence that the $L_2$ gain from $u_{lin}$ to $z$ is less than $\gamma = \sqrt{\mu}$ and a suitable choice of $F$ is given by $F = LQ^{-1}$.

Note from the $\begin{bmatrix} -\mu I & -I \\ * & -I \end{bmatrix}$ term of this LMI we can see that, as anticipated earlier, the $L_2$ gain can be no less than unity, which is achieved for the IMC scheme.
4.3 Optimisation for robustness and performance

The primary goal of anti-windup compensation is to provide performance improvement during saturation, but optimising the LMI (15) alone does not guarantee this. Indeed, there is little point in optimising (15) when an optimal solution can be found by inspection as the IMC anti-windup solution. The real use of (15) and the arguments of the previous subsection is to use them in conjunction with performance optimisation, the goal being to optimise performance and robustness together, although there will often be a trade-off.

In [15] it was argued that $\mathcal{T}_p$, the map from $u_{lin}$ to $y_d$ was central to the “true goal” of anti-windup compensation: if the induced norm of this operator was minimised, the deviation of the system’s nonlinear behaviour during and after saturation would be minimised. The paper [15] solved this problem with $\Delta_G = 0$, in the $\mathcal{L}_2$ sense, for static and low order compensators (similar treatments for discrete-time AW compensators were given in [13, 5]).

More realistically, we would really like to optimise some weighted combination of $\mathcal{T}_p$ and $\mathcal{T}_r$. This can be accomplished by solving the LMI.

\[
\begin{bmatrix}
QA_p' + A_pQ + L'B_p' + B_pL & B_pU - L' & 0 & QC_p' + L'D_p' & L' \\
* & -2U & I & UD_p' & U \\
* & * & -\mu I & 0 & -I \\
* & * & * & -W_p^{-1} & 0 \\
* & * & * & * & -W_r^{-1}
\end{bmatrix} < 0 \tag{16}
\]

in the variables $Q > 0, U = \text{diag}(v_1, \ldots, v_m) > 0, L, \mu > 0$. $W_p$ and $W_r$ are weighting matrices which reflect the relative importance of performance and robustness respectively and are chosen by the designer. The derivation of this LMI is carried out in a similar way to that of the previous section in the spirit of that done in [15].

**Remark 1:** Throughout this paper, we have only discussed full-order anti-windup compensation for two reasons:

1. A full-order anti-windup compensator always exists.

2. The expressions and derivations of formulae for static and low-order anti-windup compensators are more complex. The same types of ideas are certainly applicable to these types of compensator but, for simplicity, we do not discuss them here.

**Remark 2:** Another advantage of using LMI (16) to synthesise full-order compensators is that it tends to prevent fast poles appearing in the compensator dynamics. If a robustness weight ($W_r$) was not included in the optimisation - or if $W_r$ was only chosen small - the poles of the anti-windup compensator tend to be rather fast, lying far to the left of the imaginary axis. Obviously this would require a very high sampling frequency for implementation, which is not always possible in practice. However, when simultaneously optimising performance and robustness using (16), the poles are placed in regions more comparable to that of the controller. This feature is reminiscent
of solving ‘singular’ $\mathcal{H}_\infty$ problems with LMI’s, where poles tend to get placed far from the imaginary axis. Of course in the Riccati-based methods these singular problems are not solvable, which normally prevents the appearance of these large poles.

4.4 Stability robustness of the work in [13]

The work in [13] and [15] advocates only the optimisation of anti-windup performance, that is the minimisation of the $\mathcal{L}_2$ gain of the operator $\mathcal{P}_p$. By setting $W_r = 0$ and solving the LMI (16) we obtain a full-order compensator which only optimises this performance. This approach is the continuous time counterpart to the discrete-time approach of [13] and as argued in [15, 13], this operator is central to obtaining desirable anti-windup behaviour. It is interesting to examine whether this approach has any intrinsic robustness properties.

Suppose now that we consider output multiplicative uncertainty instead of additive uncertainty, that is $G(s) = (I + \Delta_o(s))G_2(s)$, or equivalently that $\Delta_G = \Delta_oG_2$. In this case, our expression for $y_{lin}$ becomes

$$y_{lin} = G_1d + G_2u_{lin} + \Delta_oG_2(I - M\mathcal{F}(u_{lin}))u_{lin}$$

(17)

We are sure that the system is robustly stable when $\mathcal{F}(u_{lin}) = 0$ as this is a property of the nominal linear system. Therefore the smaller we can make the extra term $-\Delta_oG_2M\mathcal{F}(u_{lin})$ the closer to nominal robustness we shall be. We can do nothing about $\Delta_o$, so the logical approach is to make

$$\|G_2M\mathcal{F}(u_{lin})\|_{i,2} = \|N\mathcal{F}(u_{lin})\|_{i,2}$$

(18)

as small as possible. Note that as in [13], because $G_2(s) = N(s)M^{-1}(s)$ is a right coprime factorisation of the plant $G_2(s)$ the quantity in equation (18) is exactly our performance operator norm $\|\mathcal{P}_p\|_{i,2}$. Therefore the minimisation of $\|\mathcal{P}_p\|_{i,2}$ leads not only to desirable anti-windup performance, but it also endows the saturated system with some indirect robustness when the the uncertainty is of the output multiplicative type. Note however, that the robustness is not guaranteed to approach that of the linear system. Nevertheless, it does appear to explain some of the results of [6] where a discrete-time version of the results of [15] were implemented on a hard-disk system. In that work, few robustness problems were encountered and the above analysis goes some way to explaining this.

5 Example

To demonstrate the implications of our results we use an example introduced in [18]. The example consists of a plant with a large resonant peak and the controller used is a two-degree-of-freedom controller with large
feedback gain. However, we shall take this plant to be the *perturbed* plant, \( \tilde{G}(s) \) rather than our nominal plant. We shall also use a controller with a slightly lower gain, for reasons which shall become clear later.

5.1 The unperturbed system

For our nominal plant we take the example of [18] without the resonant peak (i.e. the system is critically damped). Thus, \( G_2(s) \sim (A_p, B_p, C_p, D_p) \) is described by the state-space matrices:

\[
A_p = \begin{bmatrix} 0 & 1 \\ -10 & -10 \end{bmatrix} \quad B_p = \begin{bmatrix} 0 \\ 10 \end{bmatrix} \quad C_p = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad D_p = [0]
\] (19)

The linear controller \( K(s) = [K_1(s) \quad K_2(s)] \sim (A_c, [B_{cr} \quad B_c], C_c, [D_{cr} \quad D_c]) \), which was designed for the plant \( G(s) \), is described by the state-space matrices

\[
A_c = \begin{bmatrix} -80 & 0 & 2.5 \\ 1 & 0 & 0 \\ 0 & 0 & -2.5 \end{bmatrix} \quad B_{cr} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad B_c = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \quad C_c = \begin{bmatrix} -9450 & 3375 & 337.5 \end{bmatrix} \quad D_{cr} = [0] \quad D_{cy} = [-135]
\] (20)

This is the same controller as in [18] but with a lower gain in the feedback loop. The dashed line in Figure 7 shows the response of the linear system to a pulse input of magnitude 1.2. As can be seen the system is well behaved with no overshoot and a fast settling time. When the control input is saturated at \( \pm 1 \) however, the system degrades to that shown by the solid line in Figure 7. Although the system still exhibits no overshoot, the actuator saturation has impaired the system’s ability to track its reference signal accurately and there appears to be a phase lag between the reference and the output.

![Figure 7: Response of unperturbed system with no anti-windup](image-url)
To improve the behaviour of the system, static anti-windup compensation as suggested in [15] is introduced. This anti-windup compensation minimises the difference between the nominal linear system’s response and the saturated system’s response. In terms of $\gamma$, $\gamma$ was given as $\gamma \approx 24$ and a static anti-windup compensator achieving this bound was computed as

$$
\Theta = \begin{bmatrix}
-0.1909 \\
0.1402
\end{bmatrix}
$$

A diagram of how this compensator is implemented is given in Figure 8. Note that the unperturbed system is quadratically stabilisable by static anti-windup compensation, but we cannot be sure that the perturbed system also has this desirable property (in fact it does not). Figure 9 shows the response of the system with static anti-windup and it can be seen that the performance of the system has improved: the system output is now in-phase with the reference demand, although the infeasibility of the reference means it is not possible for the output to track the input with the correct magnitude. The robust and full order anti-windup compensators introduced in the following sections both yield a similar response to that in Figure 9 when used on the nominal system $G(s)$. 

Figure 8: Structure of static anti-windup scheme

Figure 9: Response of unperturbed system with optimal static anti-windup
5.2 The perturbed system

The true, or perturbed, plant, \( \tilde{G}(s) = G(s) + \Delta_G(s) \) is the plant given in [18]. This has a large resonant peak and is described by the following state-space matrices:

\[
\tilde{A}_p = \begin{bmatrix} 0 & 1 \\ -10 & -0.01 \end{bmatrix} \quad \tilde{B}_p = \begin{bmatrix} 0 \\ 10 \end{bmatrix} \quad \tilde{C}_p = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \tilde{D}_p = [0]
\]

(22)

![Figure 10: Response of perturbed system with optimal static anti-windup](image)

The dashed line in Figure 10 shows the response of this perturbed plant using the same controller as before; the controller yields a similar type of performance to before and hence can be considered satisfactory. However, when input saturation is introduced, the static anti-windup compensator actually drives the system unstable as depicted by the solid line in Figure 10. In fact, this static anti-windup is worse than no anti-windup at all, which at least remains stable. Note that for this perturbed plant and controller, static anti-windup is not feasible, so we cannot expect it to stabilise the system in question.

To overcome this problem, we choose \( W_p = 0.001 \) and \( W_r = 1 \) and we synthesise a robust dynamic compensator according to the LMI (16). This yields the matrix \( F \) as

\[
F = \begin{bmatrix} 0.2242 & 0.0446 \end{bmatrix} \times 10^{-4}, \quad \gamma \approx 1
\]

(23)

and a \( \gamma \approx 1 \). In this case we have essentially the IMC solution, as \( F \) is almost zero. As \( \gamma \approx 1 \) we can expect to recover the robustness results of the linear system. Indeed 11 shows the system’s response and we can see that the system is stable and although the response is not a good as the unperturbed system, it is substantially better than that of the static anti-windup compensation. Note that this robust anti-windup compensation also performed as well as the optimal static anti-windup compensation when applied to the unperturbed plant.
5.3 Other anti-windup compensators

As discussed in Section 4.4 the full-order anti-windup compensation method obtained by setting $W_r = 0$ and solving the LMI (16) can, in a certain sense, provide a robust anti-windup solution. This type of anti-windup solution is the continuous time counterpart of the discrete-time full-order compensator described in [13]. We designed a full-order anti-windup compensator according to this approach, choosing $W_p = 1$ and $W_r = 0.0001 \approx 0$ and then solving the LMI (16). The optimal gain matrix $F$ was given by

$$F = \begin{bmatrix} -1.3138 & -0.1424 \times 10^4 \end{bmatrix}, \quad \gamma \approx 1$$  \hspace{1cm} (24)$$

It is important to realise that in this case the guaranteed robustness margin of the system is now given by $\sqrt{\mu/W_r} \approx 100$, although this appears to be a conservative estimate. Again note that this places the poles of
the anti-windup far into the left half plane and, therefore, this would require a fast sampling frequency for correct implementation. Figure 12 shows the response of the saturated system using this compensator and the stable response is indicative of the scheme’s intrinsic robustness properties, although it does appear to be more oscillatory than that of the robust anti-windup compensator.

![Figure 12: Response of the saturated system using the compensator](image1)

Figure 12: Response of the saturated system using the compensator.

For purposes of comparison and due to its industrial popularity, we also tested a high gain anti-windup compensator ([1, 7]). In terms of Figure 8, we chose $\Theta = [0 \quad 14]'$. No formal performance or stability guarantees accompany this scheme. Nevertheless, the response of the unperturbed system using this anti-windup scheme is very similar to the response with the optimal static scheme (Figure 9). When this compensator was tested on the perturbed plant (Figure 13) stability was also obtained although note the very oscillatory response which may be indicative of poor robustness. Its response was certainly not as good as that of the robust anti-windup compensator shown in Figure 11, although it was obviously superior to that of the optimal static scheme shown in Figure 10.

![Figure 13: Response of perturbed system with high-gain anti-windup](image2)

Figure 13: Response of perturbed system with high-gain anti-windup.

For purposes of comparison and due to its industrial popularity, we also tested a high gain anti-windup compensator ([1, 7]). In terms of Figure 8, we chose $\Theta = [0 \quad 14]'$. No formal performance or stability guarantees accompany this scheme. Nevertheless, the response of the unperturbed system using this anti-windup scheme is very similar to the response with the optimal static scheme (Figure 9). When this compensator was tested on the perturbed plant (Figure 13) stability was also obtained although note the very oscillatory response which may be indicative of poor robustness. Its response was certainly not as good as that of the robust anti-windup compensator shown in Figure 11, although it was obviously superior to that of the optimal static scheme shown in Figure 10.

6 Conclusion

This paper has introduced a framework for synthesising robust anti-windup compensators for open-loop systems subject to additive uncertainties. The problem was posed in a similar way to that of linear $H_\infty$ control theory and the solution which was proposed appears as a set of LMI’s of a similar type to those proposed in [15]. The attractive feature of the proposed solution is that the class of uncertainties considered are those which are routinely considered by control practitioners, which underlines the practical relevance of the results. As an important aside, we have also demonstrated the optimal robustness of the much denigrated IMC anti-windup strategy.

We note that many of the simple anti-windup schemes also seem to be quite robust in practice. For example
the Hanus scheme ([4]) has been the practitioners method of choice for some time (the papers [7, 10] contain examples) although it is not so easy to prove this theoretically as the Hanus scheme is only globally stable for a small class of systems, so proving that it is globally robustly stable, generally, is impossible. However, it seems likely that the Hanus scheme could be examined with respect to something which might be described as local robust stability, which could constitute future work.

References


