Guaranteed stability regions of linear systems with actuator saturation using the low-and-high gain technique

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Abstract

The stability regions of input-saturated linear systems are examined, with attention focused on state-feedback LQ control. The results describe hyper-ellipsoidal domains, which are subsets of the actual domain of attraction for the system. This is not new; the novelty is in the use of the low-and-high gain technique, to reduce conservatism in the results. In addition to the standard regulator, the tracking case is treated and some comments are made regarding other extensions.

1 Introduction

Over the past decade the low gain technique and its relation, the low-and-high gain (LHG) technique, have made a significant impact in the theory of the control of systems subject to input constraints. Although its introduction in (Lin & Saberi1993) was fairly modest, it has since been used extensively in nonlinear tracking (Lin et al. (1998)), output regulation (Lin et al. (1995)) and piecewise linear control (De Dona et al. (1999),De Dona et al. (2000)). Also, a recent paper has appeared (Angeli et al. (2000)), inspired by the pseudo-scheduling ideas involved in the LHG technique but based on predictive control.

The low-and-high gain technique was introduced in Lin & Saberi (1995) as a method for improving the transient response of linear systems subject to actuator saturation, while simultaneously ensuring that the region in which stability was guaranteed did not shrink. With the work in Lin & Saberi (1993) serving as a foundation, it was shown that, for globally null controllable linear systems ¹ subject to actuator saturation, this technique allowed semi-global stabilisation to be achieved with purely linear feedback. Furthermore, the simple saturated linear feedback which resulted was appealing from a practical point of view.

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¹ A linear system is globally null controllable if it is stabilisable and has all of its eigenvalues in the closed left-half complex plane.
In Saberi et al. (1996), this work was continued further and several other interesting features of the LHG technique were discussed. In particular, the work in Lin & Saberi (1995) was extended to include general linear systems (the original work was devoted to single-input integrator chains) and the technique was shown to be robust against certain types of matched input additive uncertainty. Although most of the paper concentrated on the semi-global stabilisation problem, and thus on globally null controllable systems, comment was also made regarding systems with exponentially unstable modes; namely that the same result could be used, but in a local rather than semi-global framework.

Also contained in the work of Saberi et al. (1996) was the advocation of the LQ type of state feedback control from which the low and high gain was constructed. This allowed straightforward construction of the feedback laws, and also enabled other conclusions to be drawn about the resulting closed-loop system. In particular, a system under LQ control has the following properties:

1. **Infinite Gain Margin.** The loop gain can be increased indefinitely without loss of stability.

2. **60 degrees Phase Margin.**

3. **6dB Gain Reduction Tolerance.** The loop gain can be halved and stability will still be maintained.

These are enviable robustness properties and, in Saberi et al. (1996), the first was proved to hold also for linear systems subject to actuator saturation, provided the initial state belonged to an *a priori* established subset of the state-space. In this paper we shall use the third property to find an ellipsoidal estimate of the region of attraction of a linear system under an arbitrary LQ control.

The basis on which our results are constructed is the low-and-high gain technique, which we use to give an ellipsoidal invariant set associated with an LQ control law. The idea is actually the converse of the original technique: in the papers of Lin & Saberi (1995) and Saberi et al. (1996) a low gain is designed first, then a high gain - based on the low gain - is constructed to improve performance. Here we consider only a one-stage design, but split the control law into low and high gain parts and use the low gain component to provide a region of guaranteed stability under saturation.

This approach can, of course, be combined with the method of Saberi et al. (1996) by the inclusion of another higher gain, after the region of attraction has been calculated; effectively this leads to a reduction in conservatism. Alternatively, it can be used to test how well a given linear feedback can perform under the saturation constraints and can be used to give an indication of whether its domain of attraction is large enough. The estimate we obtain here is less conservative than would be obtained, were the control law not split into low and high gain constituents (we compare this method of estimating domains of attraction to other possibilities later in the paper).

The systems which we deal with are, at first, perfectly-known regulators with LQ control laws; later this class is extended to include set-point tracking. Brief comment is also made regarding uncertain systems, where the
robustness guarantees are shown to be quite conservative, and regarding an extension of the method described here for even less conservative results. There is another possible extension which we do not consider: the output regulation setting, which appears in the LHG context in the papers by Lin et al. (1995) and Stoorvogel & Saberi (1999). We anticipate the extension of these results should be relatively systematic for such systems.

2 Stability regions for saturated LQ regulators

2.1 The low-and-high-gain technique

Consider the stabilisable finite-dimensional linear time-invariant system

\[ \dot{x} = Ax + Bs\text{sat}(u) \]  

(1)

where \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, (A, B) \) is stabilisable, and

\[ \text{sat}(u) = \begin{bmatrix}
    \text{sat}_1(u_1) \\
    \text{sat}_2(u_2) \\
    \vdots \\
    \text{sat}_m(u_m)
\end{bmatrix} \]  

(2)

where \( \text{sat}_i(u_i) = \text{sign}(u_i) \times \min(|u_i|, \bar{u}_i) \) and \( \bar{u}_i > 0, \quad \forall i = 1, \ldots, m \) is the saturation limit of the \( i \)’th element of the control vector, \( u \). We assume the control law is given by

\[ u = -R^{-1}B^TPx \]  

(3)

and \( P > 0 \) is the positive definite stabilising solution to the ARE

\[ A^TP + PA - PB\text{diag}(R) + Q = 0 \]  

(4)

where \( R \) is diagonal \(^2\) and the matrix \( Q > 0 \). This is similar to the low-gain control proposed in Saberi et al. (1996), where \( R = I \) and \( Q = \varepsilon I \), where \( \varepsilon > 0 \) is referred to as the low gain parameter. The control \( u_L = -R^{-1}B^Px \) is referred to as the low gain control law and can be used to provide semi-global stability for asymptotically null controllable systems by choosing \( \varepsilon \) small enough.

The LHG strategy is obtained by adding to this low gain control, a high gain obtained from the solution to the ARE (4) of the form

\(^2\text{R} \) is required to be diagonal to ensure that the gain-reduction margin holds; this property is not valid if \( R \) is non-diagonal.
The composite control \( u = u_L + u_H \) will also provide semi-global stability for asymptotically null controllable linear systems regardless of the value of \( \rho \); that is \( \rho \) can be made arbitrarily large and stability will be maintained in the (arbitrarily large) region stabilised by \( u_L \). This proves that the infinite gain margin property observed in conventional LQ control also holds for asymptotically null controllable linear systems with constrained inputs.

### 2.2 Main Results

We consider the type of systems set out above but assume \( R = \text{diag}(r_1, \ldots, r_m) > 0 \) and that \( Q \) is any positive definite matrix. This is slightly more general than considered in Saberi et al. (1996) : as we are considering local stability the assumption that \( Q = \epsilon I \) is not needed and the assumption that \( R = I \) made in that paper is also not needed. Unlike standard LQ control we assume that \( R \) is diagonal to ensure that gain reduction margin holds. Our main result is

**Theorem 1** Consider the system (1) under the control (3). Then the system is asymptotically stable for all states in \( \mathcal{E} \), where

\[
\mathcal{E} = \{ x : x'Px \leq c \} \tag{6}
\]

and \( c \) is computed as

\[
c = \min_i \frac{4u_i^2 r_i^2}{B_i'PB_i} \tag{7}
\]

**Proof:** In order to apply LHG techniques we re-write the control as

\[
u = u_L + u_H \tag{8}
\]

\[
= - \frac{1}{\alpha} R^{-1} B' P x - \frac{1}{\beta} R^{-1} B' P x \tag{9}
\]

where \( \alpha^{-1} + \beta^{-1} = 1 \) and \( \alpha, \beta > 0 \). Note that the ARE (4) can be re-written as

\[
(A - \frac{1}{\alpha} B R^{-1} B')' P + P (A - \frac{1}{\alpha} B R^{-1} B') + \left( \frac{1}{\alpha} - \frac{1}{\beta} \right) P B R^{-1} B' P + Q = 0 \tag{10}
\]

Labelling \( A_c(\alpha) := A - \frac{1}{\alpha} B R^{-1} B' P \), this can be written
It is easy to see, by a Lyapunov argument, that choosing \( \alpha^{-1} \geq \beta^{-1} \) ensures \( A_c(\alpha) \) is Hurwitz. Now, from the fact that LQ control allows a 6 dB gain reduction tolerance, we choose, as a special case, \( \alpha = 2 \) (which implies \( \beta = 2 \)); then the ARE (11) becomes

\[
A_c(\alpha)^t P + PA_c(\alpha) + Q = 0
\]  

To apply the LHG techniques we must find a region of the state-space, \( \Omega \), such that \( \forall x \in \Omega \Rightarrow |u_L| \leq \bar{u} \), where \(|\cdot|\) denotes component-wise magnitude and \( \leq \) denotes component-wise inequality. A computationally convenient (but possibly conservative) method of doing this it to find

\[
c := \max x^t Px \Rightarrow |u_L| \leq \bar{u} \quad \forall i \in \{1, \ldots, m\}
\]  

This actually has an analytic solution, which can be found, for example, in the papers of Henrion et al. (1999) and Gutman & Hagander (1985):

\[
c = \min_i \frac{\bar{u}_i^2}{F_i^t P^(-1) F_i}
\]

where the control is given by \( u = Fx \). Substituting our low-gain LQ control, \( u_L \), and \( \alpha = 2 \) in this equation gives

\[
c = \min_i \frac{4\bar{u}_i^2}{(R^{-1}B^t P^1 P^{-1}(R^{-1}B^t P^1))_i}
\]

which reduces to the expression for \( c \) in the theorem. It thus remains for us to prove that the set \( \mathcal{E} \) is indeed invariant for the control law, \( u \). In closed loop, the system becomes

\[
\dot{x} = Ax + Bu_L - Bu_L + Bsat(u_L + u_H)
\]

\[
= A_c(\alpha)x + B[sat(u_L + u_H) - u_L]
\]

Choose as a Lyapunov function candidate \( v(x) = x^t Px \) and consider its derivative in the set \( \mathcal{E} \).

\[
\dot{v}(x) = -x^t Qx + 2x^t PB[sat(u_L + u_H) - u_L]
\]

\[
= -x^t Qx - 2\bar{u}_i R[sat(u_L + u_H) - u_L]
\]

\[
= -x^t Qx - 2 \sum_{i=1}^m u_L_i r_i [sat_i(u_L + u_H) - u_{L,i}]
\]
Stability follows if the last term on the right hand side is negative definite. The proof thus follows along similar lines to that in Turner et al. (2000) or Saberi et al. (1996). Note that if saturation does not occur, for some \( i \), then

\[
-2u_{L,i}r_i[sat_i(u_L + u_H) - u_{L,i}] = -2u_{L,i}r_i u_{H,i}
\]

(21)

\[
= -2u_{L,i}r_i \rho u_{L,i}
\]

(22)

\[
< 0
\]

(23)

where the last inequality follows from the fact that \( r_i, \rho > 0 \), where \( r_i \) is the \( i \)th diagonal entry of \( R > 0 \) and \( \rho \) is defined as \( \rho := \frac{\alpha}{\beta} = 1 \).

Next consider the case where the \( i \)th channel saturates i.e. \( sat_i(u_L + u_H) = \tilde{u}_i \). Note that as \( |u_{L,i}| \leq \tilde{u}_i \forall x \in \mathcal{E} \), then this implies that \( u_{H,i} \geq 0 \), which in turn implies that \( u_{L,i} > 0 = a_i \) (as \( u_H = \rho u_L \)). Note also that \( \forall x \in \mathcal{E} \)

\[
sat_i(u_L + u_H) - u_{L,i} = \tilde{u}_i - u_{L,i} \geq 0 = b_i
\]

(24)

Hence it follows that

\[
-2u_{L,i}r_i[sat_i(u_L + u_H) - u_{L,i}] = -2a_i r_i b_i \leq 0
\]

(25)

A similar result can be obtained when the \( i \)th channel saturates at its lower limit. Therefore, we can conclude that

\[
-2 \sum_{i=1}^{m} u_{L,i}r_i[sat_i(u_L + u_H) - u_{L,i}] \leq 0
\]

(26)

and hence that \( \dot{v}(x) < 0 \forall x \in \mathcal{E} \). This therefore proves that \( \mathcal{E} \) defined in the theorem is a region of attraction for the system under the LQ control law.

The basic idea of Theorem 1 is to split the LQ control law into two parts and to use the low-gain part to compute a region of attraction for the system. In contrast to the normal procedure (as used in Lin & Saberi (1995) and Saberi et al. (1996)), which calculates the region of attraction based on the initial control law never saturating, here the low gain part of the control law is not allowed to saturate, but the overall control law is allowed to.

Note here that, as \( \alpha = \beta = 2 \), then \( u_L = u_H \); we use different notation for each component of the control law as, later in the paper, this will not necessarily be the case (see Section 4.3).

This technique can be used either to calculate a region of attraction for a given LQ control law; or it can be used to calculate a less conservative estimate of the region of attraction for a system under LQ control and then a further high gain can be used to achieve better performance.

\( ^3 \)In a way this is similar to the work of Lauvdal & Fossen (1997), except they use a one-step procedure.
Remark 1: Here we only consider LQ control laws, in keeping with the convention adopted by Saberi et al. (1996) and continued in later papers. However it should be noted that, in certain circumstances for a given non-LQ control law, an inverse optimal control problem can be solved to determine the equivalent $P, Q, R$ (see Boyd et al. (1994) for a convenient LMI formulation). Thus our results are more widely applicable than they initially appear.

Remark 2: This type of ellipsoidal subset of the region of attraction becomes especially conservative for systems which have some null-controllable modes, as the region of attraction in this case is arbitrarily large in some directions (see Lauvdal & Fossen (1997) or Hu et al. (2000)). However, following similar ideas to these papers, one can separate the null-controllable and exponentially unstable modes and use this technique purely for calculating the region of attraction in the subspace corresponding to the exponentially unstable modes.

3 Guaranteed Regions for Asymptotic Tracking

In order to cope with tracking requirements, we use the co-ordinate change as given in Turner et al. (2000), which was inspired by a less general co-ordinate change given in Lin et al. (1998). Here we consider the strictly proper, stabilisable, observable system with $B$ having full column rank.

\[
\begin{align*}
\dot{x} &= Ax + B \text{sat}(u) \\
y &= Cx \\
u &= Fx + Gr
\end{align*}
\]

where $C \in \mathbb{R}^{p \times n}$, $F \in \mathbb{R}^{m \times n}$, $G \in \mathbb{R}^{m \times p}$ and $r \in \mathbb{R}^p$ is the constant set-point to be tracked. To enable the results of Turner et al. (2000) to be applied, we assume that the system has the following form (which can be computed in the MATLAB environment by, for example, the algorithm given in Edwards & Spurgeon (1998)):

\[
\begin{align*}
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\
A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\
x_2 \end{bmatrix} + \begin{bmatrix} 0 \\
\bar{B} \end{bmatrix} \text{sat}(u) \\
y &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\
x_2 \end{bmatrix} \\
u &= \begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} x_1 \\
x_2 \end{bmatrix} + Gr
\end{align*}
\]

where $\bar{B} \in \mathbb{R}^{m \times m}$ and $A_{11} \in \mathbb{R}^{(n-m) \times (n-m)}$ are nonsingular (see appendix for a discussion about the legitimacy of this). Following Lemma 1 of Turner et al. (2000), we make the co-ordinate transformation.
\[
\begin{bmatrix}
\tilde{x}_1 \\
\tilde{x}_2
\end{bmatrix} = \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} - \begin{bmatrix}
H_1 \\
H_2
\end{bmatrix} r = x - \tilde{H}r
\]  

(33)

and choose

\[
H_1 = -A_{11}^{-1}A_{12}H_2
\]  

(34)

\[
H_2 = (C_2 - C_1A_{11}^{-1}A_{12})^R
\]  

(35)

\[
G = -\tilde{B}^{-1}[(A_{21} + \tilde{B}F_1)H_1 + (A_{22} + \tilde{B}F_2)H_2]
\]  

(36)

where \((.)^R\) denotes right inverse, if it exists, and \(F\) is chosen as \(F = -R^{-1}B'Px\), where \(P > 0\) solves the ARE

\[
A'P + PA - PBR^{-1}B'P + C'QC = 0
\]  

(37)

where \(Q,R > 0\) and \(R\) is diagonal.

It has been proved in Lemma 2 of Turner et al. (2000), that for these choices of parameters, asymptotic tracking can be achieved providing saturation is not encountered. The aim here is to calculate a region of asymptotic tracking for the system under this control.

Note that \(G\) has a certain structure, in terms of \(H_1\) and \(H_2\), which initially might lead one to believe that asymptotic tracking results can only be applied to feedforward gains exhibiting this special structure. In fact this is not the case: any feedforward gain \(G\), which achieves asymptotic tracking, in the control law \(u = Fx + Gr\) satisfies the definition given above. This follows because any \(G\) which achieves asymptotic tracking has to be the inverse - or right inverse - of the system with state feedback. The next observation proves this formally.

**Observation 1** Consider the system (30) - (32) and assume \(A_{11}\) is invertible and \(C_2 - C_1A_{11}^{-1}A_{12}\) has full row rank. Then any \(G\) which achieves asymptotic tracking also satisfies the definition in (36).

**Proof:** We show that any \(G\) which achieves asymptotic tracking can be reduced to the form in (36). First note that the steady state gain of the plant with state feedback is

\[
P_{ss} = \lim_{s \to 0} C(sI - A)^{-1}B = -[C_1 \quad C_2'] 
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} + \tilde{B}F_1 & A_{22} + \tilde{B}F_2
\end{bmatrix}^{-1} \begin{bmatrix}
0 \\
\tilde{B}
\end{bmatrix}
\]  

(38)

Thus for tracking in the steady state we must have \(G = P_{ss}^R\). For convenience, define \(A_{21} + \tilde{B}F_1 := M\) and \(A_{22} + \tilde{B}F_2 := N\). Then using the matrix inversion lemma we have

\[4\] A right inverse exists if the matrix in question has full row rank - in this case it requires the reasonable assumption that the number of outputs to be controlled is less than or equal to the number of plant inputs.
\[ P_{ss} = -[C_1 \quad C_2] \left[ \begin{array}{c} -A_{11}^{-1} A_{12} (N - MA_{11}^{-1} A_{12})^{-1} \tilde{B} \\ (N - MA_{11}^{-1} A_{12})^{-1} \tilde{B} \end{array} \right] \quad (39) \]

\[ = +C_1 A_{11}^{-1} A_{12} (N - MA_{11}^{-1} A_{12})^{-1} \tilde{B} - C_2 (N - MA_{11}^{-1} A_{12})^{-1} \tilde{B} \quad (40) \]

\[ = -(C_2 - C_1 A_{11}^{-1} A_{12}) (N - MA_{11}^{-1} A_{12})^{-1} \tilde{B} \quad (41) \]

Thus it follows that \( G = P_{ss}^R \) is given by

\[ G = -\{(C_2 - C_1 A_{11}^{-1} A_{12}) (N - MA_{11}^{-1} A_{12})^{-1} \tilde{B}\}^R \quad (42) \]

\[ = -B^{-1} \{A_{22} + BF_2 - (A_{21} + BF_1) A_{11}^{-1} A_{12}\} (C_2 - C_1 A_{11}^{-1} A_{12})^R \quad (43) \]

which is the same as given in equation (36).

Thus any \( G \) which achieves asymptotic tracking can also be chosen to have the special structure given in (36), providing \( A_{11} \) is nonsingular.

**Theorem 2**  Consider the system described by (30)-(32) with \( A_{11} \) nonsingular, \( F = -R^{-1}B^T P \), \( P > 0 \) the unique positive definite solution to (37), and \( \tilde{H}, H_1, H_2 \) and \( G \) chosen as in equations (33)-(36).

Define the sets

\[ \mathcal{E} := \{ \tilde{x} : \tilde{x}^T P \tilde{x} \leq c \} \quad (44) \]

\[ \mathcal{Y}_r := \{ r : |F \tilde{r}| \leq \Delta \tilde{u} \} \quad (45) \]

where \( c \) is computed as

\[ c = \min_i \frac{4(1 - \Delta)^2 \tilde{u}_i^2 \tilde{r}_i^2}{B_i^T P B_i} \quad (46) \]

and \( \tilde{x} = x - \tilde{H} r \) and \( \tilde{J} := (F \tilde{H} + G) \).

Then for all \((\tilde{x}, r) \in \Gamma := \mathcal{E} \times \mathcal{Y}_r\) and all \( \Delta \in [0, 1] \) the output \( y \) asymptotically tracks the constant input \( r \).

**Proof:** Note that the control can be partitioned as

\[ u = u_L + u_H + \tilde{J} r \quad (47) \]

\[ := \frac{1}{\alpha} F \tilde{x} + \frac{1}{\beta} F \tilde{x} + \tilde{J} r \quad (48) \]
In accordance with the gain reduction tolerance of 6dB property of the LQ control law, let $\alpha = 2$ (which also implies that $\beta = 2$). This means that the ARE (37) can be written as

$$A_c(\alpha)'P + PA_c(\alpha) + C'QC = 0$$

(49)

Next we have to compute a region of “attraction”, which is more involved as it depends on the set-point value as well as the state. To apply the LHG technique, we must have that $|u_L + \tilde{J}r| \leq \tilde{u}, \forall (\tilde{x}, r) \in \Gamma$, where $\Gamma$ is some compact set to be defined. Let us compute $\mathcal{S}^\prime$ as in expression (44) where $c$ is given as

$$c := \max \tilde{x}P\tilde{x} \Rightarrow \frac{1}{\alpha}F\tilde{x} \leq (1 - \Delta)\tilde{u}$$

(50)

As $\alpha = 2$, $c$ can be calculated as the analytic solution of this maximisation problem:

$$c = \min_i \frac{4(1 - \Delta)^2\tilde{u}_i^2}{(R^{-1}B^TP)^{-1}(R^{-1}B^P)^i}$$

(51)

which reduces to the expression given in the theorem. Let us compute $\mathcal{V}_r$ as

$$\mathcal{V}_r := \{ r : |\tilde{J}r| \leq \Delta\tilde{u} \}$$

(52)

Defining $\Gamma := \mathcal{S} \times \mathcal{V}_r$ it is easy to see that

$$|u_L + \tilde{J}r| \leq \frac{1}{\alpha}F\tilde{x} + |\tilde{J}r| \leq |\tilde{u}| \quad \forall (\tilde{x}, r) \in \Gamma$$

(53)

Next, we must show that $\forall (\tilde{x}, r) \in \Gamma$, the closed loop system is stable. In closed loop we have, in the $\tilde{x}$ coordinates,

$$\dot{\tilde{x}} = A\tilde{x} + A\tilde{H}r + B_{sat}(u_L + u_H + \tilde{J}r)$$

$$= A_c(\alpha)\tilde{x} + B_{sat}(u_L + u_H + \tilde{J}r) - u_L - \tilde{J}r$$

(54)

(55)

Choosing $v(\tilde{x}) = \tilde{x}P\tilde{x} > 0$ as a Lyapunov function yields

$$\dot{v}(\tilde{x}) = \tilde{x}'(A_c(\alpha)'P + PA_c(\alpha))\tilde{x} + 2\tilde{x}'PB_{sat}(u_L + u_H + \tilde{J}r) - u_L - \tilde{J}r$$

$$= -\tilde{x}'C'QC\tilde{x} + 2\tilde{x}'PB_{sat}(u_L + u_H + \tilde{J}r) - u_L - \tilde{J}r$$

(56)

(57)

In the same way as the proof for Theorem 1 it can be shown that, for all $(\tilde{x}, r) \in \Gamma$,.
However as $C^TQC$ is, in general only positive semi-definite, we have yet to prove asymptotic stability of the origin $\tilde{x} = 0$. First note that as both right-hand side terms in equation (57) are negative semi-definite, for the resulting equation to be equal to zero, both must be zero independently. First consider the terms in (58) and note that for this to be zero either $Px \in \mathcal{N}(B')$ or

$$\text{sat}(u_L + u_H + \tilde{J}r) - u_L - \tilde{J}r = 0$$

(59)

For this to hold, no saturation must take place and this implies that $u_L = u_H = 0$. This and $Px \in \mathcal{N}(B')$ both imply that the system remains open-loop, as $u_L = 0$; hence its dynamics are governed by $\dot{\tilde{x}} = A\tilde{x}$.

Next note that for the first right-hand term in (57) to be zero, we must have $\sqrt{Q}C\tilde{x} = 0$. Differentiating this $(n-1)$ times gives

$$\begin{bmatrix}
\sqrt{Q}C \\
\sqrt{Q}CA \\
\ldots \\
\sqrt{Q}CA^{n-1}
\end{bmatrix} \tilde{x} = 0$$

(60)

Note that as we have assumed $(A,C)$ observable, this implies that $\tilde{x} = 0$ and also implies that $A_{\alpha}(\alpha)$ is Hurwitz, by equation (49). Hence $\nu(\tilde{x}) = 0$ only at $\tilde{x} = 0$. This implies that $\mathcal{E}$ is a region of attraction in the $\tilde{x}$ co-ordinates. As $\lim_{t \to \infty} y(t) = \lim_{t \to \infty} C(\tilde{x}(t) + \tilde{H}r) = r$, this implies asymptotic tracking in the original co-ordinates for all $(x - \tilde{H}r, r) \in \mathcal{E} \times \mathcal{Y}_r$.

\textbf{Remark 3:} The role of the scalar parameter $\Delta$ is to balance the trade-off between the size of set-point which can be tracked with the size of the ellipsoid of initial conditions (Lin et al. (1998)). For a small $\Delta$, the size of the set-point, determined by the polyhedral $\mathcal{Y}_r$, is small, but the size of $\mathcal{E}$ is large. Note that if $\Delta = 0$ then the set $\mathcal{Y}_r$ only consists of the point $r = 0$ and hence the problem reduces to that of the regulator. For a larger $\Delta$, the size of reference able to be tracked is increased, but the size of $\mathcal{E}$ is consequently reduced. \hfill $\Box$ \hfill $\Box$

4 Other Extensions

Thus far we have considered the computation of regions of attraction for saturated linear quadratic regulators and similar regions for asymptotic tracking in linear systems under LQ-type control. There are other possible areas of constrained control where these LHG techniques can be applied.
4.1 Output Regulation

In Stoorvogel & Saberi (1999) and Lin et al. (1995), the concept of output regulation was discussed in the context of LHG design for saturated systems. This is the situation whereby the output of an input saturated linear system is required to track the output of an exo-system. The situation is not as straightforward as the state regulation problem, but it was shown in Lin et al. (1995) that it can be tackled with the same LHG approach. The converse type of approach we discuss here should be fairly simple to extend to such situations.

4.2 Uncertain Regulation

Consider the system with input additive uncertainty

\[ \dot{x} = Ax + B_{\text{Sat}}(u + g(x)) \]  

(61)

where \(\|g(x)\| \leq g_o(\|x\|) + D_o, g_o(.) \) locally Lipschitz and \(g_o(0) = 0\).

In Saberi et al. (1996) it was shown how the LHG technique can be used to construct feedback laws which provide either a region of asymptotic stability or a set of ultimate boundedness for such systems - depending on whether the uncertainty has a constant term in it. For these systems the freedom in constructing such feedbacks was given by the ability to choose \(\rho\) arbitrarily large. Essentially, ultimate boundedness was guaranteed by choosing \(\rho\) sufficiently large.

Here we do not have the luxury of choosing \(\rho\) infinitely large; \(\rho = \alpha \beta^{-1}\) is fixed as unity. It is thus difficult to guarantee ultimate boundedness or stability in the face of uncertainty. Following the analysis of Saberi et al. (1996), for stability to be guaranteed we must choose

\[ \rho \geq \max(\rho_1^*, \rho_2^*) \]  

(62)

where

\[ \rho_1^* := \frac{16N\max(P)M^2 \lambda_{\max}(P)}{\lambda_{\min}(Q) \lambda_{\min}(P)} \]  

(63)

\[ \rho_2^* := \frac{16N\max(P) D^2_0 \lambda_{\max}(P) c}{\lambda_{\min}(Q)} \]  

(64)

where \(M = \sup_{x \in [0,F]} \left\{ \frac{g_o(x)}{x} \right\}, F = c \sqrt{\lambda_{\min}(P^{-1})}, N = \max_{x \in [0,D_o+MF]} \delta(2s)\) and \(\delta(.)\) is a continuous map depending on the saturation function - see Saberi et al. (1996) for details.

Frequently the terms on the right hand side will be larger than unity (our fixed value of \(\rho\)) and thus the results here will not guarantee a domain of attraction for a given uncertainty. Alternatively, the maximum amount of
uncertainty tolerable for a given domain of attraction could be calculated, although typically this would be small as $\rho$ is fixed as unity.

Remark 4: It is important to emphasize that the above comments do not imply that a perturbation $g(x)$ cannot be tolerated by the system. Rather they suggest that it is difficult to prove stability in the manner proposed here without introducing some conservatism.

4.3 Further reducing conservatism

Recall from the proof of Theorem 1 that we chose $\alpha = 2$ to enlarge the ellipsoidal estimate of the region of attraction. In fact, in some cases, it is possible to go further than this and choose $\alpha > 2$.

From the proof of Theorem 1, we want to enforce

$$ A'(\alpha)P + PA_c(\alpha) = -(\alpha^{-1} - \beta^{-1})PBR^{-1}B'P - Q < 0 \quad (65) $$

Of course choosing $\alpha = 2$ accomplishes this in a simple way. Noting that $\alpha^{-1} + \beta^{-1} = 1$, and defining $\mu := \alpha^{-1}$, we can see that our condition can be re-written as

$$ (2\mu - 1)PBR^{-1}B'P + Q > 0 \quad (66) $$

It is obvious that we wish to minimise $\mu$ (maximise $\alpha$) to obtain the largest estimate of the domain of attraction. Now we will show that choosing $\mu \geq \frac{1}{2} \min \{1, \mu^*\}$, will result in such an inequality holding. First assume that $(2\mu - 1) \geq 0$, or alternatively $\mu \geq \frac{1}{2}$. To minimise $\mu$ we thus have to choose it as $\mu = \frac{1}{2}$, which as before ensures that inequality (66) holds. Next assume the other alternative, that is $(2\mu - 1) < 0$, or, equivalently, $\mu < \frac{1}{2}$. In this case we have

$$ (2\mu - 1)PBR^{-1}B'P + Q \geq (2\mu - 1)\|PBR^{-1}B'P\|I + Q \quad (67) $$

$$ \geq (2\mu - 1)\|PBR^{-1}B'P\|I + \lambda_{\min}(Q)I \quad (68) $$

The right-hand side of (68) will be greater than zero if

$$ \mu > \frac{1}{2}(1 - \frac{\lambda_{\min}(Q)}{\|PBR^{-1}B'P\|}) =: 2\mu^* \quad (69) $$

Thus if $\mu$ is chosen such that $\mu \geq \frac{1}{2} \min \{1, \mu^*\}$, then (66) will hold. If both $Q$ and $PBR^{-1}B'P$ are nonsingular, (69) can be strengthened to
\[ \mu > \frac{1}{2}(1 - \frac{1}{\lambda_{\text{min}}(PBR^{-1}B'PQ^{-1})}) \]  

(70)

Alternatively, (66) can be solved as a convex optimisation problem in \( \mu \) (generally \( PBR^{-1}B'P \) is singular). This is particularly useful if \( Q \) is rank deficient, which is not the case in the regulator problem, but when this less conservative approach is adapted to the tracking case the term \( C'QC \) is used instead of \( Q \). Thus as \( C \) will commonly be deficient in column rank, even if \( Q > 0 \), the product \( C'QC \) will only be positive semi-definite. In this case the expression (69) does not offer any improvement over Theorem 2 as \( \lambda_{\text{min}}(Q) = 0 \), yielding \( \mu = \frac{1}{2} \Rightarrow \alpha = 2 \). LMI optimisation can therefore be used to circumvent this problem, albeit at an increase in computational burden.

5 Comparison to other techniques

The technique we have advocated here is, in our opinion, useful as it enables a simple computation of an ellipsoidal domain of attraction for a given LQ control law. This can be used to evaluate the performance of a pre-designed LQ control law under saturation.

This technique is still conservative however as an ellipsoid may be a poor approximation of a saturated system’s region of attraction. If a highly accurate estimation is required a polyhedral estimate based on linear programming - such as the one proposed in Gilbert & Tan (1991) or those referred to in the work of Blanchini (1999) - may be preferable. On the negative side, many of these polyhedral based estimates are computationally expensive, and struggle for high dimensional systems. Similarly, the technique will also prove conservative when compared to descriptions of domains of attraction given in Hu et al. (2000) - again the computation is quite expensive.

When compared to other, simpler techniques, the LHG approach is quite competitive. For example, particularly with systems containing unstable modes, a crude estimate of the region of attraction (as used by, for example Gutman & Hagander (1985) and Wredenhagen & Belanger (1994)) could be given by calculating

\[ \mathcal{E}_c = \{ x : x'Px \leq c_c \} \]  

(71)

where \( c_c \) is computed as

\[ c_c = \min_i \frac{\bar{u}_i^2}{(R^{-1}B'P,B'P)^{-1}R^{-1}(B'P)^{1/2}} \]

(72)

Note that \( c_c \) is a quarter of the size of the \( c \) given by Theorem 1, implying that the guaranteed domain of attraction given by Theorem 1 is significantly larger than given in this estimate.

\[ ^5\text{We do not explicitly describe this adaption, as it should be clear from this and the previous section how this is done.} \]
Other ellipsoidal estimates have appeared which are based on LMI’s, where optimisation is used to enlarge the ellipsoid as much as possible. A particularly good estimate using these techniques has appeared in Theorem 6 of Hindi & Boyd (1998) (see also Pittet et al. (1997) for related work), where a Popov type analysis is used to obtain LMI conditions for calculating the domain of attraction. However, while being attractive for systems of low state-dimension (see next section) and a low number of control inputs, this positive appeal is depleted for higher dimensional systems with multiple control inputs, due to the computational complexity involved. Furthermore, the optimisation contains \( m \) free parameters, \( r_i \geq 1 \), which represent the amount by which the actuators are permitted to “over-saturate” (see De Dona et al. (2000)). The ellipsoidal approximates calculated for different \( r_i \) can be vastly different; for too large \( r_i \)’s the LMI’s can also be infeasible. Hence, as these \( r_i \)’s appear in a non-convex manner, a manual search generally needs to be performed to find the most suitable values.

Predictive control techniques, such as those of Angeli et al. (2000), can also be used to treat this and similar problems. The main drawback with such methods is, again, the computational burden. To quote from Remark 3 in the above paper: “The computational burden of the overall procedure is in general rather high...”, due in part to the \( \rho \)-parameterised ARE and the potentially large quadratic programming problem which needs to be solved.

### 5.1 Theorem 1 and the work of Lauvdal et al.

Apart from the paper of Saberi et al. (1996), the work to which this paper is perhaps most closely related, is that by Lauvdal & Fossen (1997) who consider the stabilisation of linear unstable systems. In fact in Lemma 4 of the above paper, a region of attraction is proposed which turns out to be a special case of Theorem 1 (although the derivation of this domain is carried out somewhat differently). Specifically, Lauvdal & Fossen (1997) propose a ball of attraction as

\[
\mathcal{B}_u = \left\{ z : \|z\| \leq 2 \min_i \bar{u}_i \right\}
\]

in the co-ordinates \( z = P_2^z x \), where \( P = P_1^z P_2^z \). In the \( x \) co-ordinates this is the ellipsoid \( \mathcal{E} \) proposed in Theorem 1 with \( R \) fixed as the identity. The results we give in section 4.3 show how to increase this domain of attraction further. The crucial conceptual difference in the work here and that of Lauvdal & Fossen (1997), is that we split an existing control into two and then use the low-high-gain control method, whereas the work of Lauvdal & Fossen (1997) is a one-step methodology.

### 6 Examples

To illustrate the results we use two simple, low dimensional examples. Low dimensionality can give a coloured view of the results, as generally, ellipsoidal estimates are not as conservative for systems of such dimensions. However, just as the conservatism of ellipsoids increases with dimensionality, the computational demands of
more accurate methods also rapidly increase - possibly reducing the plausibility of such approaches for high dimensional systems.

6.1 Regulation

To begin with, the results of Theorem 1 are applied to a simple two dimensional, single-input linear system:

\[
A = \begin{bmatrix} 0.5 & 1 \\ 0 & 0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

(74)

with control constraint \(|u| \leq 1\). Note that the system has two exponentially unstable modes, so we can expect its domain of attraction to be bounded in both dimensions. Using the LQ design method with \(Q = I_2\) and \(R = 1\), the control law was computed, yielding \(P = \begin{bmatrix} 4.6131 & 2.3692 \\ 2.3692 & 2.9471 \end{bmatrix} > 0\). Using Theorem 1, \(c\) was calculated as \(c = 1.3573\). For comparison, a crude ellipsoidal estimate using equation (72) was also computed, giving \(c_c = c/4 = 0.3393\).

Figure 1 shows a comparison between the system’s actual domain of attraction, the ellipsoidal estimate proposed in this paper, and the crude ellipsoidal estimate. Notice that the method proposed here gives a significantly less conservative estimate of the system’s domain of attraction than the crude estimate allows. It is, however, some way short of the actual domain of attraction, although this can be expected, due to the far greater computation involved with computing this. The reduced-conservatism method contained in Section 4.3 was not included as it gave only a slightly larger ellipsoid than that given by Theorem 1.

Figure 1: Domain of attraction and estimates for unstable system

Figure 2 shows a comparison between estimates provided by the method proposed in this paper and the estimate advocated in Hindi & Boyd (1998) for two different values of the free parameter \(r\) \(^6\). Note that for \(r = 1.1\), the

\(^6\)As we are dealing with a SISO system, there is only one free parameter
area of the ellipse of Hindi & Boyd (1998) is less than the area of the ellipse proposed in this paper, but for 
\( r = 2.5 \), the area of the ellipse is considerably larger. However it is interesting to note that even for such a simple

system, the LMI optimisation procedure required to calculate such an ellipsoid involved over \( 4.78 \times 10^5 \) floating

point operations - plus a manual search over \( r \). Noting the polynomial-time complexity of most LMI-solving

algorithms and problems of numerical ill-conditioning, it can thus be concluded that, for higher order MIMO

systems, this method can run into difficulties.

Figure 2: Domain of attraction and LMI estimates for unstable system

6.2 Tracking

To demonstrate the results of Theorem 2, we consider the simple second order model of the short-period longi-
tudinal axis of a fighter aircraft, taken from (Barbu et al.1999).

\[
A = \begin{bmatrix}
-1 & 1 \\
15 & -3
\end{bmatrix} \quad B = \begin{bmatrix}
0 \\
18
\end{bmatrix} \quad C = [1 \quad 0] \quad D = 0
\]

The system is subject to the control constraint \(|u| \leq 0.35\). It has one exponentially unstable mode, so we can

expect its region of attraction to be bounded in one direction, but to reach infinity in the other. As we are using

ellipsoidal estimates (which are, by their nature, compact sets) we can expect, at the outset conservative results.

Taking \( \Delta = 0.7, Q = 10 \) and \( R = 1 \) we obtain the bound on the reference as \(|r| \leq 0.3675\), where \( H_1 = 1, H_2 = 1 \)

have been designed according to Theorem 2. \( G = 3.2318 \) was taken as the inverse of the closed-loop plant’s d.c.
gain. Next we compute the region of attraction for \( \tilde{x} \) - which in the orignal \( x \) co-ordinates gives us the region of

asymptotic tracking. The ellipsoid is parametrised by \( P = \begin{bmatrix}
1.9862 & 0.1903 \\
0.1903 & 0.0262
\end{bmatrix} \) > 0 and \( c = 0.0052 \). The crude

estimate was given as \( c_c = c/4 = 0.0013 \).

Figure 3 shows the estimates of the region of asymptotic regulation in the \( x \)-plane. The middle ellipsoid is the

estimate given by Theorem 2; the inner is that given by the crude estimate. Notice that the estimate proposed
here is less conservative than the crude one. However, both estimates are extremely conservative compared with the actual region of asymptotic tracking which is not shown here (due to its comparatively large size). Also shown is the ellipsoid obtained using LMI optimisation (the formula (69) could not be used as \(\text{rank}(C'QC)\) was not full), which is slightly larger than that obtained via Theorem 2: \(c\) was found as 0.007 using this method. Note that the use of LMI’s for high-dimensional systems must be balanced against the large increase in computation required: the small increase in the size of the guaranteed region of attraction may not be worth the corresponding computational complexity.

![Figure 3: Ellipsoidal estimates of domains of asymptotic tracking](image)

7 Conclusion

This note has presented a simple technique for calculating regions in which stability is guaranteed for arbitrary LQ control laws. Both the regulation and the tracking situations have been considered and it has been shown how this technique is less conservative than the crude estimate which is sometimes used.

The appealing aspect of the proposal here is that it is computationally un-intensive, although it does suffer from conservatism when compared to other more computationally intensive procedures for estimating a system’s domain of attraction. In this respect, the results may be seen as preliminary; future research will attempt to reduce the conservatism without destroying the method’s simplicity. This problem is particularly challenging for practical systems which suffer from complex dynamics and therefore high dimensionality.

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A Nonsingularity Issues

We seek to prove the following claim:

Claim 1 Consider

\[
\dot{x} = Ax + Bu, \quad A \in \mathbb{R}^{n \times n}, \ B \in \mathbb{R}^{n \times m}
\] (76)

If \((A, B)\) is stabilisable and \(B\) has full column rank, there exists a state similarity transformation such that an equivalent representation of such a system can be found as

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} +
\begin{bmatrix}
0 \\
\bar{B}
\end{bmatrix} u
\] (77)

with \(A_{11} \in \mathbb{R}^{(n-m) \times (n-m)}\) and \(\bar{B} \in \mathbb{R}^{m \times m}\) nonsingular.
**Proof:** First note that a realisation as in eq. (77) can be found for any system using QR decomposition - see Edwards & Spurgeon (1998) for details. Furthermore, this algorithm ensures that $\bar{B}$ is nonsingular. It thus remains to prove that $A_{11}$ can be guaranteed nonsingular.

Note that as $(A, B)$ has been assumed stabilisable, it follows that

$$\text{rank} \left( \begin{bmatrix} A_{11} - vI & A_{12} \\ A_{21} & A_{22} - vI \end{bmatrix} \right) = n \quad \forall v \geq 0 \quad (78)$$

As $\bar{B}$ is nonsingular it is evident that

$$\text{rank} ([A_{11} - vI, A_{12}, 0]) = n - m \quad \forall v \geq 0 \quad (79)$$

Or equivalently that $(A_{11}, A_{12})$ must be stabilisable, which implies $\exists T : \max_i \Re(\lambda_i(A_{11} - A_{12}T)) < 0$. Thus introducing the transformation

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ T & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (80)$$

where $T$ is such that $A_{11} - A_{12}T$ is Hurwitz, in these co-ordinates we thus have:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} A_{11} - A_{12}T & A_{12} \\ T(A_{11} - A_{12}T) + A_{21} - A_{22}T & TA_{12} + A_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \bar{B} \end{bmatrix} u \quad (81)$$

where $A_{11} - A_{12}T$ is invertible by virtue of it being Hurwitz. This completes the proof. ✽✽