Abstract

The use of linear quadratic theory in bumpless transfer is discussed and a modification to a similar scheme proposed in [9] is suggested. Formulae are given for a feedback element which minimises a cost function deemed appropriate for ensuring bumpless transfer. The formulae are algebraic and depend only on given matrices and the solution to a single Riccati equation. The differences between this modified scheme and the original one are highlighted and it is suggested that, at the expense of slightly increasing the dimension of the compensator, the new version of the formulae could perform better than the previous scheme.

1 Introduction

As linear controllers give their best performance around a certain operating point of a nonlinear plant, it is not uncommon for several different linear controllers to be designed around different operating points. One controller is then switched to another when the system enters a regime where the current controller no longer gives satisfactory performance. With this strategy it is hoped that the system will behave, in some way, optimally due to the appropriate linear controller being used near its design operating point.

The process of switching, however, can severely degrade transient performance and hence schemes have been sought which enable ‘bumpless transfer’ between controllers. Classically, a high gain feedback loop has been placed around the off-line controller in order to force the off-line control signal to approach that of the on-line one ([5]). One of the major developments in this area was made by Hanus ([6]), who proposed the so-called ‘conditioning scheme’; this has had a great deal of attention devoted to it and it has also been applied industrially ([7]). Since then, a number of other schemes have been suggested; in particular some attempts to unify existing schemes and treat stability more rigorously have been made in [2] and [3]. A fairly comprehensive study of most of these schemes, along with a new $H^\infty$ based approach, can be found in [4].

In this paper we consider the idea of linear quadratic bumpless transfer, which was originally suggested in [9]. This method makes few additional assumptions other than the controllers are finite-dimensional linear time invariant (FDLTI), controllable and observable. Thus the method can be applied to quite general types of problems, but, through the use of two constant weighting matrices, does give some flexibility in design. For the sake of brevity, this paper will only consider the continuous time case, but analogous results for the discrete time case can also be derived, see [9].

2 Notation and Assumptions

The notation used throughout this paper is, in the main, standard. In the interests of clarity and simplicity we shall, wherever possible, omit the dimensions of vectors and matrices, unless it is important to draw the reader’s attention to them.

We shall assume that all of the controllers are FDLTI and that their states are available to the designer: a modest assumption since most modern controllers will be realised in software form, so the states will be computer variables. Importantly, we make the assumption that all controllers’ realisations are completely controllable and observable and locally stabilise the plant in question. We shall furthermore assume that all signals available to, and produced by, the on-line controller are available. Effectively this means the off-line controller has access to the on-line control signal, the plant output, the error signal and the reference signal.

The Lebesgue space of all continuous square-integrable functions is given by:

$$\mathcal{L}_2 = \left\{ x : \int_0^\infty x'x dt < \infty \right\}$$

where $x = x(t) \in \mathbb{R}^n$

The Extended Lebesgue space of continuous square integrable functions, $\mathcal{L}_{2e}$, is identical to the Lebesgue space, $\mathcal{L}_2$, except that the upper limit of the integral is taken as some finite time, so that functions which have an infinite $\mathcal{L}_2$ norm may still belong to $\mathcal{L}_{2e}$. In this paper it is assumed that all
signals belong to $L_2$. 

This paper considers only strictly proper 2 D.O.F controllers\(^1\), for the sake of brevity, which are assigned the following state space realisations:

\[
\begin{align*}
\dot{x} &= Ax + B_1 r + B_2 y \\
u &= Cx
\end{align*}
\]

Unless otherwise stated, the state vector is $x \in \mathbb{R}^n$, the control signal is $u \in \mathbb{R}^m$, the output of the plant is $y \in \mathbb{R}^p$, and the reference signal is $r \in \mathbb{R}^p$. An arbitrary exogenous vector is sometime used and is represented as $w \in \mathbb{R}^q$.

### 3 Linear Quadratic Bumpless Transfer

The aim of this section is to introduce the basic idea behind linear quadratic bumpless transfer, although a full discussion is beyond the scope of this paper (the interested reader should consult [9] and references therein).

In any bumpless transfer scheme, it might be expected that if the signal produced by the off-line controller is the same as, or close to, in some sense, that produced by the on-line controller, when the two controllers are switched, no bump during transfer will occur. Thus we seek a feedback gain, $F$, which can be used to drive the off-line controller to produce the same signal as the on-line controller. This type of configuration is shown in Figure 1, where the feedback matrix drives the off-line controller, and has access to the controller state as well as other external signals in the loop.

To achieve a ‘minimal’ amount of transient behaviour during switching we choose to minimise a quadratic cost function. The weighted combination of two signals is proposed.

To pose this problem in the LQ context, we minimise the following functional,

\[
J(u, \alpha, T) = \frac{1}{2} \int_0^T \left( z_u(t)'W_u z_u(t) + z_\alpha(t)'W_\alpha z_\alpha(t) \right) dt + \frac{1}{2} \tilde{z}_u(T)'P \tilde{z}_u(T)
\]

where

\[
\begin{align*}
z_u(t) &= u(t) - \tilde{u}(t) \\
z_\alpha(t) &= \alpha(t) - r(t)
\end{align*}
\]

and where $\tilde{u}(t)$ and $r(t)$ are the on-line control signal and reference signal respectively; $u(t)$ is the off-line control signal; $\alpha(t)$ is the signal produced by the feedback gain and which drives the off-line controller. $W_u$ and $W_\alpha$ are constant positive definite weighting matrices of appropriate dimension which are used to tailor the design as required. Finally, $z_u(T) = u(T) - \tilde{u}(T)$ is the difference between the two control signals at the terminal time $T$ (which will most commonly be taken as infinity), and $P$ is the positive definite terminal weighting matrix; although it may well be set to zero and is introduced only for the benefit of derivation, and, indeed, generality.

To synthesise a feedback matrix, $F$, we can therefore solve the problem of minimising this quadratic performance index. The signal $\alpha(t)$ produced by $F$, is a function of the on-line controller states, the reference and output signals, and the on-line control signal. The gain $F$ can be regarded as a full-information ‘sub-controller’, which temporarily controls the off-line controller.

To derive $F$ we invoke some standard LQ procedures, details of which can be found in [8]. If the off-line controller is being driven by the signal $\alpha(t)$, then its state space equations are,

\[
\begin{align*}
\dot{x} &= Ax + B_1 \alpha + B_2 y \\
u &= Cx
\end{align*}
\]

Substituting for $u$, in the performance index 4, we obtain:

\[
J = \frac{1}{2} \int_0^T (Cz - \tilde{u})'W_u(Cz - \tilde{u}) +
\]
Forming the Hamiltonian,

\[ H = \frac{1}{2}[(Cx - \bar{u})'W_u(Cx - \bar{u}) + (\alpha - \tau)'W_e(\alpha - \tau)] + \lambda'(Ax + B_1\alpha + B_2y) \]

(10)

From here, the first order necessary conditions are obtained as:

\[
\begin{align*}
\frac{\partial H}{\partial \lambda} &= Ax + B_1\alpha + B_2y \\
\Rightarrow \dot{\lambda} &= Ax + B_1\alpha + B_2y \\
\frac{\partial H}{\partial x} &= A'\lambda + C'W_uCx - C'W_u\bar{u} \\
\Rightarrow \lambda &= -A'\lambda - C'W_uCx + C'W_u\bar{u} \\
\frac{\partial H}{\partial \alpha} &= W_e\alpha + B_1\lambda - W_ee \\
\Rightarrow \alpha &= -W_e^{-1}B_1\lambda + \tau \\
\end{align*}
\]

(12)

If we now use this expression for \( \alpha \) in the state and co-state equations, 13 and 15 respectively, we obtain:

\[
\begin{bmatrix}
\dot{x} \\
\dot{\lambda}
\end{bmatrix} =
\begin{bmatrix}
A & -R \\
-Q & -A'
\end{bmatrix}
\begin{bmatrix}
x \\
\lambda
\end{bmatrix} +
\begin{bmatrix}
B_1 & B_2
\end{bmatrix}
\begin{bmatrix}
\tau \\
y
\end{bmatrix} +
\begin{bmatrix}
0 \\
C'W_u
\end{bmatrix}
\bar{u}
\]

(17)

where \( R = B_1W_e^{-1}B_1' \) and \( Q = C'W_uC \).

The above non-homogenous differential equation is of the form which often arises in LQ minimisation and can be solved by the Method of Sweep (see [8]). Therefore we have,

\[
\lambda(t) = \Pi(t)x(t) - g(t) \\
\]

(18)

Differentiating,

\[
\dot{\lambda}(t) = \dot{\Pi}(t)x(t) + \Pi(t)\dot{x}(t) - \dot{g}(t) \\
\]

(19)

If we now combine 18, 19 and 17 we can derive two expressions for \( \lambda \) and equate coefficients to obtain the Riccati differential equation:

\[
\dot{\Pi} = \Pi A + A'\Pi - \Pi RB_1 + Q \\
\]

(20)

A problem now exists: for implementation, we require the co-state vector \( \lambda \), which is obtained from a differential equation which develops backward in time from the terminal values of \( \bar{u}, r, \) and \( y \): we require future knowledge in order to arrive at a solution. For this reason, the formulae are normally used in their infinite horizon forms, where \( F \) becomes dependent purely on an algebraic Riccati equation (ARE), and other known signals.

Under the assumptions given in [9], the expression for \( \alpha \) in the infinite horizon is given by:

\[
\alpha = F \begin{bmatrix} x \\ y \\ \bar{u} \end{bmatrix} \\
\]

(23)

where,

\[
F = -W_e^{-1}\begin{bmatrix} (B_1'\Pi)' \\ -(B_1'(A - R\Pi)^{-T}\Pi B_2)' \\ -(W_e + B_1'(A - R\Pi)^{-T}\Pi B_1)' \end{bmatrix}^T \\
\]

(24)

and \( \Pi \) is the positive semi-definite stabilising solution to the ARE:

\[
\Pi A + A'\Pi - \Pi RB_1 + Q = 0 \\
\]

(25)

Thus we have a constant matrix, \( F \), which can be computed off-line, to enable bumpless transfer. Note also that the off-line control loop will be stable as \( \Pi \) is the stabilising solution to the ARE.

4 A Modification

The method just discussed is a useful technique as it makes few assumptions on the controllers, and in the infinite horizon case, \( F \) is purely static, meaning little extra on-line computation is needed during implementation. However, a possible criticism which could be levelled at this technique, one which does not apply to the Hanus scheme, is that there is a discontinuity at the controller input during switching. Even though at the plant input, assuming the two control signals is purely static, meaning little extra on-line computation is needed during implementation. However, a possible criticism which could be levelled at this technique, one which does not apply to the Hanus scheme, is that there is a discontinuity at the controller input during switching. Even though at the plant input, assuming the two control signals are similar, there is a negligible discontinuity, the signal driving the controller will switch from \( \alpha \) to \( \tau \) at the time of transfer.

Although, simulation results have shown that this does not have a great influence on the system’s transient responses, in certain circumstances it could, and therefore is desirable to remove. For this reason the configuration in Figure 2 is proposed.

The scheme is a slight modification of the previous one, except the off-line controller is driven by \( \alpha \) and \( \tau \) simultaneously. Note that \( \bar{\alpha} \) is \( \alpha \), which is derived differently here,
fed through a filter, \( L(s) \). When the controller is switched on-line, the switch between \( F \) and \( L(s) \) is opened, thus disconnecting \( L(s) \) from \( \alpha \). However, as the controller is driven by \( \alpha \), and as \( L(s) \) is a continuous linear system, \( \alpha \) will die away gradually, and thus the controller input will not be subject to a discontinuity.

Generally, \( L(s) \) will be chosen as a unity gain low pass filter so that \( \alpha \) is identical to \( \dot{y} \) in every respect, except that its high frequency components will be removed. This will mean the same cost function can be used, and also no discontinuity will be present at the controller input.

We consider the same cost function, 4 as before and assign \( L(s) \) the following minimal state space realisation:

\[
\begin{align*}
\dot{x}_1 &= A_1 x_1 + B_1 \alpha \\
\dot{\alpha} &= C_1 x_1
\end{align*}
\]

These dynamics are then appended to those of the off-line controller to obtain:

\[
\begin{align*}
\dot{x} &= \bar{A} \tilde{x} + B_1 w + B_2 \alpha \\
\dot{u} &= \bar{C} \tilde{x}
\end{align*}
\]

where,

\[
\begin{align*}
\bar{A} &= \begin{bmatrix} A & B_1 C_1 \\ 0 & A_1 \end{bmatrix} \\
\bar{B}_1 &= \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \\
\bar{B}_2 &= \begin{bmatrix} 0 \\ B_1 \end{bmatrix} \\
\bar{C} &= \begin{bmatrix} C \\ 0 \end{bmatrix}
\end{align*}
\]

The cost function can now be written as:

\[
\begin{align*}
\frac{1}{2} \int_0^T (\bar{C} \tilde{x} - \bar{u})' W_u (\bar{C} \tilde{x} - \bar{u}) + \alpha' W_e \alpha \, dt \\
+ \frac{1}{2} \tilde{z}_u(T)' P \tilde{z}_u(T)
\end{align*}
\]

Thus the Hamiltonian becomes,

\[
H = \frac{1}{2} \left( (\bar{C} \tilde{x} - \bar{u})' W_u (\bar{C} \tilde{x} - \bar{u}) + \alpha' W_e \alpha \right)
+ \lambda' (\bar{A} \tilde{x} + \bar{B}_1 \dot{w} + \bar{B}_2 \alpha)
\]

Evaluating the first-order necessary conditions as before leads to:

\[
\begin{align*}
\dot{\bar{z}} &= \bar{A} \tilde{x} - \bar{Q} \bar{z} \\
\bar{z} &= \begin{bmatrix} 0 \\ \bar{C} \tilde{x} \end{bmatrix}
\end{align*}
\]

where:

\[
\bar{R} := \bar{B}_2 W_e^{-1} \bar{B}_2
\]

\[
\bar{Q} := \bar{C} W_u \bar{C}
\]

Equation 33 can be combined with the Method of Sweep, and then the resulting formulae extended to the infinite horizon. Extensive algebra yields the following expression for \( \alpha \):

\[
\alpha = F \begin{bmatrix} \tilde{x} \\ w \end{bmatrix}
\]

where \( F \) is given by,

\[
\begin{align*}
-W_e^{-1} \bar{B}_2 [(-\bar{A} + \bar{R})^{-1} \Pi \Pi_b]^T \\
((\bar{A} - \bar{R})^{-1} \bar{C} W_u)^T
\end{align*}
\]

and \( \Pi \) is the solution to the ARE:

\[
\bar{A}' \Pi + \Pi \bar{A} - \Pi \bar{R} \Pi + \bar{Q} = 0
\]

5 Example

This section will illustrate the modification to the LQ bumpless transfer scheme with an example. We consider the flight of a GKN Westland Lynx helicopter as it moves from low air-speed to high air-speed. The controllers used are \( H^\infty \) 2 D.O.F controllers which have zero, non-square direct feedthrough terms.

This is an example of a highly nonlinear plant whose dynamics vary considerably as its speed increases. To obtain adequate performance two controllers are used and switched between as the helicopter makes the transition from low to high speed. Note that the original Hanus scheme cannot be applied directly, but the modified LQ scheme lends itself well to this problem.

Figure 3 shows the simulated helicopter response as its speed increases due to the pitch down manoeuvre. Notice that the helicopter performance degrades as the speed increases with the low speed controller, and how the initial response is poor with the high speed controller.

Figure 4 shows the simulated helicopter response using the modified LQ bumpless transfer scheme, as it increases speed. Transfer occurs at approximately 16 seconds, and there is only a small perceptible ’bump’ in the response, but the response remains well-behaved at all times. The weights used in the design were \( W_u = 1000 I_4 \) and \( W_e = 0.01 I_6 \).
It has been demonstrated that LQ theory can be applied successfully to the problem of bumpless transfer. Moreover, if certain modest assumptions are made, the matrix which enables this bumpless transfer is static and thus requires little extra on-line computation. The few assumptions made concerning the controller also allow the technique to be applied to most types of linear controllers. The modification introduced eliminates the discontinuity at the controller input, at the expense of introducing extra dynamics to the system.

Future research will concentrate on how the selection of the weighting matrices influences the design, and if there are any guidelines for choosing them in an optimal fashion. The stipulation of controller controllability and observability may also be investigated. Furthermore, the possibility of using an LQ scheme as an anti-windup compensator might be given consideration.

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References