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Working Paper No. 19 12
September 2019

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27th September, 2019.

Abstract

This paper re-examines the possibility of endogenous long-term economic growth in neoclassical models with non-renewable resources. Instead of using a Cobb-Douglas production function as in most existing studies, we consider a general form in which physical capital is functionally separable from labour and natural resources. It is shown that if the elasticity of substitution between labour and resources is identical to one, then long-term economic growth is endogenous. But if this elasticity is not equal to one, as suggested by empirical studies, then long-term economic growth is entirely driven by an exogenous technological factor.

Keywords: Non-Renewable Resources; Endogenous Growth; Elasticity of Substitution.

JEL classification: O13, O41, Q32.

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1 Introduction

Economists have long been concerned with the issue of natural resource scarcity and its implications on economic growth. In a seminal paper, Stiglitz (1974) provides an analysis of these issues using the now-standard neoclassical growth framework with infinitely-lived consumers. It is shown that perpetual growth in per-capita output is possible even when natural resources are limited in quantity but essential for production.[1] More importantly for the present study, the long-term growth rate in Stiglitz's model is endogenously determined. Endogenous economic growth means that the long-term growth rate is not determined *a priori* by some exogenous technological factors, but rather it is derived within the model and can be influenced by firms' and consumers' economic behaviour. This result has profound implications for both resource economics and economic growth theory, as it suggests that practices and policies in natural resource utilisation and management can affect an economy's long-term performance. In a more recent study, Agnani, Gutiérrez and Iza (2005, henceforth AGI) show that this endogenous growth result remains valid in a similar neoclassical framework but with overlapping generations of consumers.

In this paper, we re-examine the possibility of endogenous economic growth in neoclassical models with non-renewable resources. Our starting point is the observation that both Stiglitz (1974) and AGI adopt the same production function, namely the Cobb-Douglas specification with three inputs (physical capital, labour and natural resources).[2] This is essentially assuming that the elasticity of substitution between any pair of inputs is constant and equal to one. This assumption, however, is at odd with many empirical findings.[3] While the estimates produced by the empirical literature may vary across datasets and estimation methods, the general consensus is that the Cobb-Douglas specification is not empirically supported. This raises the question of whether the endogenous growth result in Stiglitz (1974) and AGI are robust to other forms of production function. The purpose of the present study is to address this question. We believe this is an important step in bridging the gap between theoretical and empirical research on this topic.

In order to single out the role of the unitary elasticity assumption, we adopt the same analytical framework as in AGI but replace their overly restrictive Cobb-Douglas production function with

[1] More specifically, perpetual growth in per-capita variables is possble in the presence of (resource-augmenting) technological improvements and a high degree of substitutability between capital input and resource input. This result is also mentioned in Jones and Manuelli (1997, p.91).

[2] By "Cobb-Douglas specification," we mean the production function is multiplicatively separable in the three inputs and has constant elasticities. This specification is commonly used in resource economics. See, for example, Solow (1974), Mitra (1983), Barbier (1999) and Groth and Schou (2002) among others.

[3] See, for instance, Kemfert (1998), Kemfert and Welsch (2000), van der Werf (2008), Henningsen, Henningsen and van der Werf (2018).

more general ones. In our benchmark model, we begin with a general class of production functions that exhibit constant returns to scale in all inputs and in which physical capital is functionally separable from labour and natural resources.[4] Similar to AGI, we focus on characterising balanced growth equilibria, i.e., competitive equilibria in which all major economic variables (such as output, capital stock and consumption) are growing at some constant rate. We show that two types of balanced growth equilibria are possible, depending on the elasticity of substitution between labour input and resource input. On the one hand, if this elasticity is constant and equal to one, then the long-term economic growth rate is endogenously determined as in AGI. This result holds regardless of the elasticity of substitution between capital input and the other two inputs. Thus, this can be viewed as a generalisation of the AGI result. But, on the other hand, if the elasticity of substitution between labour input and resource input is bounded above or below by one, then long-term economic growth is solely determined by an exogenous labour-augmenting technological factor as predicted by the standard neoclassical growth model. Taken together, our benchmark results underscore the pivotal role of the unitary elasticity assumption in generating endogenous economic growth.[5]

The economic intuition behind these results can be explained as follows: As is well-known in the economic growth literature, perpetual growth in per-capita variables is possible only in the presence of certain factors (either exogenous or endogenous) that can counteract the diminishing marginal product of physical capital.[6] These factors are dubbed as "the engine of growth." In the model of Stiglitz (1974) and AGI, total factor productivity (TFP) and resource input jointly served as the engine of growth. The crucial thing is that the utilisation rate (and hence the growth rate) of natural resources in their models is endogenously determined by factors other than the exogenous TFP. This opens up a channel through which other factors (such as consumers' preferences and shares of factor income) can affect the engine of growth and the long-term growth rate. How is this related to the unitary elasticity assumption? In any balanced growth equilibrium, the quantities of effective labour input and effective resource input are changing over time, but the shares of aggregate output distributed as labour income and expenses on resource input must remain constant.[7] This condition is automatically satisfied if the elasticity of substitution between labour input and resource

[4] The terminology and definition of "functional separability" are taken from Leontief (1947) and Blackorby and Russell (1976). Further details are provided in Section 2.

[5] In Appendix C, we show that our benchmark results can be easily extended to an environment with infinitely-lived consumers as in Stiglitz (1974). This suggests that the unitary elasticity assumption also plays a crucial role in Stiglitz's results.

[6] See Jones and Manuelli (1997, Section 2) for an in-depth discussion on this point.

[7] Effective labour input is equal to the number of labour hours multiplied by a labour-augmenting technological factor. Similarly, effective resource input means the product of resource input and a resource-augmenting technological factor. These variables are formally defined in the model section.

input is identical to one (i.e., a Cobb-Douglas function in these two inputs). For all other cases, this condition is satisfied only if the ratio between these two inputs (in effective units) is constant over time. This creates a restriction on the utilisation rate of natural resources. In particular, this rate is now solely determined by the exogenous growth rate of labour input and technological factors. As a result, the engine of growth is solely determined by the exogenous technological factors.

As a robustness check, we consider several other specifications of production function in Section 4.1. We find that the exogenous growth result remains valid in all these cases, thus further casting doubt on the generality of the endogenous growth result.

The rest of the paper is organised as follows. Section 2 describes the setup of the benchmark model. Section 3 presents the main results concerning the balanced growth equilibria of the model. Section 4 examines several alternative specifications of the production function and makes some remarks on our results. Section 5 concludes.

2 The Benchmark Model

2.1 Consumers

The model considered here is essentially the same as that in AGI, except for a more general form of production function. Unless otherwise stated, we will adopt the same notations as in AGI to facilitate comparison between the two work.

Time is discrete and is indexed by $t \in \{0, 1, 2, ...\}$. In each time period, a new generation of identical consumers is born. The size of generation t is given by $N_t = (1+n)^t$, where $n \geq 0$ is the population growth rate. Each consumer lives two periods, which we will refer to as the young age and the old age. All young consumers have one unit of time which is supplied inelastically to work. The market wage rate at time t is denoted by w_t. All consumers are retired when old. There are two types of commodities in this economy: a composite good which can be used for consumption and investment, and non-renewable natural resources which are primarily used as input of production. All prices are expressed in units of the composite good.

Consider a consumer who is born at time $t \geq 0$. Let $c_{1,t}$ and $c_{2,t+1}$ denote his young-age and old-age consumption, respectively. The consumer's lifetime utility is given by

$$U(c_{1,t}, c_{2,t+1}) \equiv \ln c_{1,t} + \frac{1}{1+\theta} \ln c_{2,t+1}, \tag{1}$$

where $\theta > 0$ is the rate of time preference. The consumer can accumulate wealth by investing in physical capital and natural resources. Let s_t and m_t denote, respectively, the consumer's holdings of physical capital and natural resources. The rate of return from physical capital is denoted by r_{t+1}, and the price of natural resources at time t is p_t.

Taking $\{w_t, r_{t+1}, p_t, p_{t+1}\}$ as given, the consumer's problem is to choose a consumption profile $\{c_{1,t}, c_{2,t+1}\}$ and an investment portfolio $\{s_t, m_t\}$ so as to maximise his lifetime utility in (1), subject to the budget constraints:

$$c_{1,t} + s_t + p_t m_t = w_t, \qquad \text{and} \qquad c_{2,t+1} = (1 + r_{t+1})\, s_t + p_{t+1} m_t. \tag{2}$$

The first-order conditions of this problem imply the following:

$$c_{2,t+1} = \left(\frac{1 + r_{t+1}}{1 + \theta}\right) c_{1,t}, \tag{3}$$

$$\frac{p_{t+1}}{p_t} = 1 + r_{t+1}. \tag{4}$$

Equation (3) is the Euler equation of consumption, which determines the growth rate of individual consumption between young and old ages. Equation (4) is the Hotelling rule, which is essentially a no-arbitrage condition. It states that in order for the consumer to invest in both types of assets, the capital gain from natural resources must be equal to the gross return from physical capital. Using (2)-(4), we can derive the optimal level of consumption,

$$c_{1,t} = \left(\frac{1 + \theta}{2 + \theta}\right) w_t \qquad \text{and} \qquad c_{2,t+1} = \left(\frac{1 + r_{t+1}}{2 + \theta}\right) w_t, \tag{5}$$

and the optimal level of investment in physical capital,

$$s_t = \frac{w_t}{2 + \theta} - p_t m_t. \tag{6}$$

2.2 Production

On the supply side of the economy, there is a large number of identical firms that produce the composite good. In each time period $t \geq 0$, each firm hires labour (N_t), rents physical capital (K_t) and purchases extracts of natural resources (X_t) from the competitive factor markets, and produces output (Y_t) according to the production technology

$$Y_t = F\left(K_t, G\left(Q_t X_t, A_t N_t\right)\right). \tag{7}$$

In the above expression, Q_t is a resource-augmenting technological factor and A_t is a labour-augmenting technological factor. Both are assumed to grow at some constant exogenous rate, so that $Q_t = (1+q)^t$ and $A_t = (1+a)^t$, with $q > 0$ and $a \geq 0$, for all $t \geq 0$.

The production function in (7) is specified as a composition of two functions, $F(\cdot)$ and $G(\cdot)$. Intuitively, one can interpret this as a two-stage production process: In the first stage, effective units of labour and natural resources are combined using an aggregator function $G(\cdot)$. The resultant is then combined with physical capital using another aggregator function $F(\cdot)$ to produce the final output. In the terminology of Leontief (1947) and Blackorby and Russell (1976, p.286), the subset of inputs $\{Q_t X_t, A_t N_t\}$ is said to be functionally separable from K_t. There is more than one way to define functional separability with three inputs. Another possibility is to assume that $\{K_t, Q_t X_t\}$ is functionally separable from $A_t N_t$. A third possibility is to assume that $\{K_t, A_t N_t\}$ is functionally separable from $Q_t X_t$. We will tackle these alternative specifications in Section 4.

The main properties of (7) are summarised in Assumptions A1 and A2. Recall that an input is said to be *essential for production* if no output can be produced without some positive amount of this input [Dasgupta and Heal (1974) and Solow (1974, p.34)] Throughout this paper, we will use $F_i(\cdot)$ to denote the partial derivative of $F(\cdot)$ with respect to its ith argument, and $F_{ij}(\cdot)$ to denote the partial derivative of $F_i(\cdot)$ with respect to its jth argument, $i, j \in \{1, 2\}$. The partial derivatives of $G(\cdot)$ are similarly represented.

Assumption A1 Both $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ and $G : \mathbb{R}^2_+ \to \mathbb{R}_+$ are twice continuously differentiable, strictly increasing, strictly concave and exhibit constant returns to scale (CRTS) in their arguments.

Assumption A2 Each input $I \in \{K, X, N\}$ is either essential for production or its marginal product is unbounded when I is arbitrarily close to zero.

Assumption A1 is a list of conditions that are commonly used in the economic growth literature. These conditions imply that the composite function in (7) is also twice continuously differentiable, strictly increasing, strictly concave and exhibits CRTS in all three inputs. In neoclassical growth models (without natural resources), it is also common to impose two other assumptions on the production function: First, both physical capital and labour are essential for production. Second, the marginal product of these inputs are unbounded as their quantity approach zero. These assumptions, however, are rather restrictive. For instance, within the class of constant-elasticity-of-substitution (CES) production functions, only Cobb-Douglas production functions satisfy both of these assumptions.[8] Our Assumption A2 gets around this problem by requiring only one of these properties to hold, and this is sufficient to ensure that in equilibrium all three inputs will be used in every time period. The arguments are as follows: As Solow (1974) suggests, it is natural and reasonable to focus on equilibria that have strictly positive output in every period. If an input is deemed essential for production, then a strictly positive amount must always be used in this kind of equilibrium. On the other hand, since both factor markets and goods markets are competitive, the price of any input must be equated to its marginal product in equilibrium. If the marginal product of an input is unbounded at or around zero, then the marginal benefit of using an infinitesimal quantity of this input will for sure outweigh the marginal cost. Hence, it is never optimal to use a zero quantity of this input.

In Appendix A, we show that Assumption A2 is satisfied by various forms of nested CES production functions, some of which have been used in empirical studies.[9] In particular, the production function in (7) can take the form of a two-stage CES function as in Sato (1967) when $F(\cdot)$ and $G(\cdot)$ are given by

$$F(K_t, Z_t) = \left[\alpha K_t^{\eta} + (1-\alpha) Z_t^{\eta}\right]^{\frac{1}{\eta}}, \qquad \text{with } \alpha \in (0,1) \text{ and } \eta < 1, \tag{8}$$

$$G(Q_t X_t, A_t N_t) \equiv \left[\varphi (Q_t X_t)^{\psi} + (1-\varphi)(A_t N_t)^{\psi}\right]^{\frac{1}{\psi}}, \qquad \text{with } \varphi \in (0,1) \text{ and } \psi < 1. \tag{9}$$

The production function in AGI corresponds to the special case in which $\eta = \psi = 0$. Under this "double Cobb-Douglas" specification, the two technological factors A_t and Q_t are observationally equivalent to a single Hicks neutral technological factor (or total factor productivity), $B_t \equiv Q_t^{v} A_t^{\beta}$. Because of this, the separate effects of A_t and Q_t are not considered in AGI.

Since the production function exhibits CRTS in all three inputs, we can focus on the profit-maximisation problem faced by a single representative firm. Let R_t be the rental price of physical capital and $\delta \in (0,1)$ be the depreciation rate. The representative firm's problem is given by

$$\max_{K_t, X_t, N_t} \left\{F\left(K_t, G\left(Q_t X_t, A_t N_t\right)\right) - R_t K_t - p_t X_t - w_t N_t\right\},$$

[8] The same point has also been made by Dasgupta and Heal (1974, p.14) and Solow (1974, p.34) in natural resource economics. Solow (1974) suggests that this is the main justification for using the Cobb-Douglas production function in his work.

[9] See the empirical studies mentioned in Footnote 3 and also the references therein.

and the first-order conditions are

$$R_t = r_t + \delta = F_1 \left(K_t, G \left(Q_t X_t, A_t N_t \right) \right), \tag{10}$$

$$p_t = Q_t F_2 \left(K_t, G \left(Q_t X_t, A_t N_t \right) \right) G_1 \left(Q_t X_t, A_t N_t \right), \tag{11}$$

$$w_t = A_t F_2 \left(K_t, G \left(Q_t X_t, A_t N_t \right) \right) G_2 \left(Q_t X_t, A_t N_t \right). \tag{12}$$

2.3 Natural Resources

The economy has a fixed and known quantity of non-renewable natural resources which can be costlessly extracted in any time period. The initial size of the stock is denoted by $M_0 > 0$.[10] Let M_t be the stock available at the beginning of time t, and X_t be the quantity extracted and sold in the market at time t.[11] Define the extraction rate (or utilisation rate) at time t as $\tau_t \equiv X_t / M_t$. The stock of natural resources then evolves according to

$$M_{t+1} = M_t - X_t = \left(1 - \tau_t \right) M_t. \tag{13}$$

2.4 Competitive Equilibrium

Given the initial conditions: $K_0 > 0$ and $M_0 > 0$, a competitive equilibrium of this economy includes sequences of allocation $\{c_{1,t}, c_{2,t+1}, s_t, m_t\}_{t=0}^{\infty}$, aggregate inputs $\{K_t, N_t, X_t\}_{t=0}^{\infty}$, natural resources $\{M_t\}_{t=0}^{\infty}$ and prices $\{w_t, R_t, p_t, r_{t+1}\}_{t=0}^{\infty}$ such that,

(i) Given prices, $\{c_{1,t}, c_{2,t+1}, s_t, m_t\}$ solves the consumer's problem in any period $t \geq 0$.

(ii) Given prices, $\{K_t, N_t, X_t\}$ solves the representative firm's problem in any period $t \geq 0$.

(iii) The stock of natural resources evolves according to (13).

(iv) All markets clear in every period, which means $K_{t+1} = N_t s_t$ and $M_{t+1} = N_t m_t$ for all $t \geq 0$.

[10] At time 0, the initial stock of physical capital and non-renewable resources are owned by a group of "initial old" consumers. The decision problem of these consumers is trivial and does not play any role in the analysis of balanced growth equilibrium.

[11] This notation is slightly different from the one in AGI. Specifically, these authors define M_t as the stock remaining at the end of time t (after extraction). This difference is immaterial since we both focus on balanced growth paths along which M_t depletes at a constant rate.

3 Main Results

Similar to AGI, we focus on balanced growth equilibria. Specifically, these are competitive equilibria that satisfy four additional conditions:

(v) Per-worker output (Y_t/N_t) grows at a constant rate $\gamma^* - 1$, for some $\gamma^* > 0$.

(vi) The ratio of physical capital to output is constant over time, i.e., $K_t = \kappa^* Y_t$, for some $\kappa^* > 0$.

(vii) The rate of return from physical capital is constant over time, i.e., $r_t = r^*$, for some $r^* > -\delta$.

(viii) The utilisation rate of non-renewable resources is positive and constant over time, i.e., $\tau_t = \tau^*$, for some $\tau^* \in (0,1)$.

Conditions (v)-(vii) are consistent with the empirical observations made by Kaldor (1963) and many subsequent studies in the economic growth literature. Conditions (v) and (vi) together imply that Y_t and K_t must be growing at the same rate in any balanced growth equilibrium, i.e.,

$$\frac{K_{t+1}}{K_t} = \frac{Y_{t+1}}{Y_t} = \gamma^* (1+n).$$

Condition (viii) is a common feature in economic growth models with natural resources. Given the simple linear structure of (13), this condition implies that X_t and M_t must be decreasing at the same constant rate in any balanced growth equilibrium, so that[12]

$$\frac{X_{t+1}}{X_t} = \frac{M_{t+1}}{M_t} = 1 - \tau^*.$$

Before proceeding further, we first review some of the most fundamental results in AGI. According to their Lemma 1 and Proposition 1, if the production function is given by

$$Y_t = B_t K_t^{\alpha} N_t^{\beta} X_t^{v},$$

where $\alpha > 0$, $\beta > 0$, $v > 0$, $\alpha + \beta + v = 1$, and B_t is a measure of total factor productivity (TFP) that grows at a constant positive rate $b > 0$, then a unique balanced growth equilibrium exists with

[12] Stiglitz (1974) and Groth and Shou (2007) are among those studies that require a constant extraction rate in balanced growth equilibrium. Scholz and Ziemes (1999), Grimaud and Rougé (2003) are two examples that consider a constant growth rate of X_t in balanced growth equilibrium.

τ^* and γ^* jointly determined by

$$\frac{\gamma^*(1+n)}{(1-\tau^*)} = \frac{\alpha(1+n)(2+\theta)\gamma^*}{\beta-(2+\theta)v(1-\tau^*)/\tau^*} + 1 - \delta, \tag{14}$$

$$\gamma^* = (1+b)^{\frac{1}{1-\alpha}}\left(\frac{1-\tau^*}{1+n}\right)^{\frac{v}{1-\alpha}}. \tag{15}$$

Once τ^* and γ^* are known, the value of r^* and κ^* are given by

$$1+r^* = \frac{\gamma^*(1+n)}{1-\tau^*} \qquad \text{and} \qquad \kappa^* = \frac{\alpha}{r^*+\delta}. \tag{16}$$

In the sequel, we will refer to this as the *AGI solution*.

The main implication of the AGI solution is that both τ^* and γ^* are *endogenously* determined by a number of factors, including the TFP growth rate (b), population growth rate (n), depreciation rate (δ), the share of factor incomes in total output (α, β and v), and the consumers' rate of time preference (θ). If we decompose B_t according to $B_t \equiv Q_t^v A_t^\beta$ and define $\widehat{k}_t \equiv K_t/(A_t N_t)$ as physical capital per effective unit of labour, and $\widehat{x}_t \equiv (Q_t X_t)/(A_t N_t)$ as effective unit of resource input, then the AGI solution also implies

$$\frac{\widehat{k}_{t+1}}{\widehat{k}_t} = \left(\frac{\widehat{x}_{t+1}}{\widehat{x}_t}\right)^{\frac{v}{1-\alpha}} = \left[\frac{(1+q)(1-\tau^*)}{(1+a)(1+n)}\right]^{\frac{v}{1-\alpha}}. \tag{17}$$

Thus, depending on the solution of (14)-(15), $\widehat{k}_t$ and $\widehat{x}_t$ can be monotonically increasing, monotonically decreasing or constant over time in the unique balanced growth equilibrium.

To highlight the significance of these findings, consider an alternate economy with $v = 0$ in AGI's production function. Natural resources are now no longer needed in the production process and, as a result, $B_t \equiv A_t^{1-\alpha}$.[13] In any balanced growth equilibrium, a constant r_t immediately implies a constant value of $\widehat{k}_t$. This in turn implies that per-worker capital and per-worker output must be growing at the same rate as A_t, so that $\gamma^* = (1+a)$.[14] This is nothing but a restatement of a well-known result: In the standard neoclassical growth model where production function exhibits CRTS in K_t and $A_t N_t$, long-term growth in per-capita variables is entirely driven by the exogenous labour-augmenting technological factor.[15]

[13] It follows immediately that $\tau_t = \tau^* = 0$ for all t. In this alternate economy, natural resources play the same role as the intrinsically worthless asset in the rational bubble model of Tirole (1983).

[14] This can also be seen by setting $\tau^* = 0$ and $v = 0$ in equations (15) and (17).

[15] This result holds in both overlapping-generation models and models with infinitely-lived consumers.

When compared to this alternate economy, the AGI solution shows that introducing productive natural resources can transform an otherwise exogenous growth model into one with endogenous growth. If, in addition, the solution of (14)-(15) satisfies $(1+q)(1-\tau^*) > (1+a)(1+n)$, then the endogenous long-term growth rate is strictly greater than $1+a$.

We now return to the question of whether the AGI solution is robust under a more general production function. Our next two theorems provide an answer to this question based on the composite function in (7). At the core of the analysis is the elasticity of substitution between the two inputs of $G(\cdot)$. This elasticity can be defined using the function $g(\widehat{x}) \equiv G(\widehat{x}, 1)$ for all $\widehat{x} \geq 0$. Under Assumption A1, $g(\cdot)$ is twice continuously differentiable with $g'(\cdot) > 0$ and $g''(\cdot) < 0$. By the CRTS property of $G(\cdot)$, we can write

$$G(QX, AN) = AN \cdot g(\widehat{x}),$$

where $\widehat{x} \equiv QX/(AN)$. As shown in Arrow *et al.* (1961) and Palivos and Karagiannis (2010), the elasticity of substitution of $G(\cdot)$ can be expressed as[16]

$$\sigma_G(\widehat{x}) = -\frac{g'(\widehat{x})}{\widehat{x}g(\widehat{x})} \frac{g(\widehat{x}) - \widehat{x}g'(\widehat{x})}{g''(\widehat{x})} > 0, \qquad \text{for all } \widehat{x} > 0. \tag{18}$$

In particular, $G(\cdot)$ is Cobb-Douglas if and only if $\sigma_G(\cdot)$ is identical to one.

Our Theorem 1 states that if $\sigma_G(\cdot)$ is identical to unity, then the long-term growth factor γ^* and the utilisation rate τ^* are determined similarly as in the AGI solution. This is true even if $F(\cdot)$ does not take the Cobb-Douglas form. This result thus provides a partial generalisation of the AGI solution. But, on the other hand, if $\sigma_G(\cdot)$ is never equal to one (which means it is either uniformly bounded above or uniformly bounded below by one), then any balanced growth equilibrium (if exists) must satisfy $\gamma^* = (1+a)$ and $(1+q)(1-\tau^*) = (1+a)(1+n)$. In other words, the AGI solution is no longer valid. This result is formally stated in our Theorem 2. The proof of these and other theoretical results can be found in Appendix B.

Theorem 1 *Suppose Assumptions A1 and A2 are satisfied and $G(\cdot)$ takes the following Cobb-Douglas form:*

$$G(Q_tX_t, A_tN_t) = (Q_tX_t)^{1-\phi}(A_tN_t)^{\phi}, \qquad \text{with } \phi \in (0,1). \tag{19}$$

[16] As explained in Arrow *et al.* (1961, p.228-229), this expression is derived under two assumptions: (i) both the factor markets and goods markets are competitive and (ii) $G(\cdot)$ exhibits CRTS. Both assumptions are satisfied in our model.

Define $b \equiv (1+a)^{\phi}(1+q)^{1-\phi} - 1$. *Then any balanced growth equilibrium (if exists) must satisfy*

$$\gamma^* = (1+b)\left(\frac{1-\tau^*}{1+n}\right)^{1-\phi}, \tag{20}$$

$$(1+r^*)(1-\tau^*) = \gamma^*(1+n), \tag{21}$$

$$\gamma^*(1+n) = \chi^* F_2(1,\chi^*)\left[\frac{\phi}{2+\theta} - \left(\frac{1-\tau^*}{\tau^*}\right)(1-\phi)\right], \tag{22}$$

$$F_1(1,\chi^*) = r^* + \delta. \tag{23}$$

Theorem 1 provides a system of equations that can be used to determine the value of four key variables in any balanced growth equilibrium (provided that such an equilibrium exists). These are the growth factor of per-worker output (γ^*), the utilisation rate of natural resources (τ^*), the rate of return from physical capital (r^*) and the ratio between $(\widehat{x}_t)^{1-\phi}$ and $\widehat{k}_t$ (denoted by χ^*). All other variables in a balanced growth equilibrium can be uniquely determined using these four values. Similar to the AGI solution, the utilisation rate τ^* must be greater than a certain threshold $\overline{\tau} \in (0,1)$. To see this, note that both $\gamma^*(1+n)$ and $\chi^* F_2(1,\chi^*)$ are strictly positive, thus it follows from (22) that

$$\frac{\phi}{2+\theta} - \left(\frac{1-\tau^*}{\tau^*}\right)(1-\phi) > 0$$

$$\Rightarrow \tau^* > \overline{\tau} \equiv \frac{(2+\theta)(1-\phi)}{\phi + (2+\theta)(1-\phi)}. \tag{24}$$

If $F(\cdot)$ also takes a Cobb-Douglas form, say $F(K_t, Z_t) = K_t^{\alpha} Z_t^{1-\alpha}$, with $\alpha \in (0,1)$, then we can get

$$\chi^* = \left(\frac{r^*+\delta}{\alpha}\right)^{\frac{1}{1-\alpha}} \qquad \text{and} \qquad \chi^* F_2(1,\chi^*) = \frac{1-\alpha}{\alpha}(r^*+\delta).$$

Upon substituting these into (22) and setting $\phi = \beta/(1-\alpha)$ and $(1-\phi) = v/(1-\alpha)$, we can obtain

$$\gamma^*(1+n) = \frac{1}{\alpha}(r^*+\delta)\left[\frac{\beta - (2+\theta)v(1-\tau^*)/\tau^*}{2+\theta}\right].$$

This, together with (21), gives us

$$\frac{\alpha(2+\theta)(1+n)\gamma^*}{\beta - (2+\theta)v(1-\tau^*)/\tau^*} = r^* + \delta = \frac{\gamma^*(1+n)}{1-\tau^*} - (1-\delta),$$

which is the same equation that appears in AGI's Lemma 1 part (i). According to their Proposition 1, a balanced growth equilibrium exists and is unique. We now show that similar results can be obtained if $F(\cdot)$ is a CES function with elasticity of substitution strictly greater than one.

Proposition 1 *Suppose $F(\cdot)$ is given by (8) with elasticity of substitution $\sigma_F \equiv (1-\eta)^{-1} \geq 1$. Suppose $G(\cdot)$ takes the Cobb-Douglas form in (19). Then the economy has at least one balanced growth equilibrium. If, in addition,*

$$\left[(1+b)\left(\frac{1+n}{1-\overline{\tau}}\right)^{\phi} - (1-\delta)\right]^{\eta} > \alpha\,(1-\eta)^{1-\eta}\,, \tag{25}$$

where $\overline{\tau}$ is the threshold level defined in (24), then a unique balanced growth equilibrium exists.

Note that the additional condition (25) is automatically satisfied when $\eta = 0$. Hence, the above proposition subsumes AGI's existence and uniqueness results as special case. Condition (25) is also readily satisfied if physical capital depreciates fully after one period, i.e., $\delta = 1$ (a common assumption in OLG models). In this case, the left side of (25) is strictly greater than one for any $b > 0$, $n > 0$, $\overline{\tau} \in (0,1)$, $\phi \in (0,1)$ and $\eta \in (0,1)$; while the right side is strictly less than one for any $\alpha \in (0,1)$ and $\eta \in (0,1)$.

It is, however, more difficult to ensure the existence and uniqueness of balanced growth equilibrium when $\sigma_F < 1$ (or equivalently, $\eta < 0$). In this case, it is possible that slight changes in σ_F will lead to drastic changes in the results. The following numerical example is intended to demonstrate this. First, we combine equations (20)-(23) to form a single equation in τ^*, which is

$$\frac{(2+\theta)(1+b)(1+n)^{\phi}(1-\tau^*)^{1-\phi}}{\phi - \left(\frac{1-\tau^*}{\tau^*}\right)(2+\theta)(1-\phi)} = \frac{r(\tau^*)+\delta}{\alpha}\left\{\left[\frac{r(\tau^*)+\delta}{\alpha}\right]^{\frac{\eta}{1-\eta}} - \alpha\right\}, \tag{26}$$

where $r(\tau^*) \equiv (1+b)(1+n)^{\phi}(1-\tau^*)^{-\phi} - 1$. We then evaluate the two sides of this equation over a range of value of τ using the following parameterisation: Suppose one model period takes 25 years. We set $\theta = 1.775$ so that the annual subjective discount factor equals to 0.96. We set the annual employment growth rate to 1.6%, which matches the average annual growth rate of U.S. employment over the period 1953-2008. This implies $n = (1.0160)^{25} - 1 = 0.4871$. The annual TFP growth rate is taken to be 1.05%, which is in line with the estimates reported by Feng and Serletis (2008, p.300). The implied value of b is 0.2984 over a 25-year period. We also set $\delta = 1$, $\phi = 0.38$

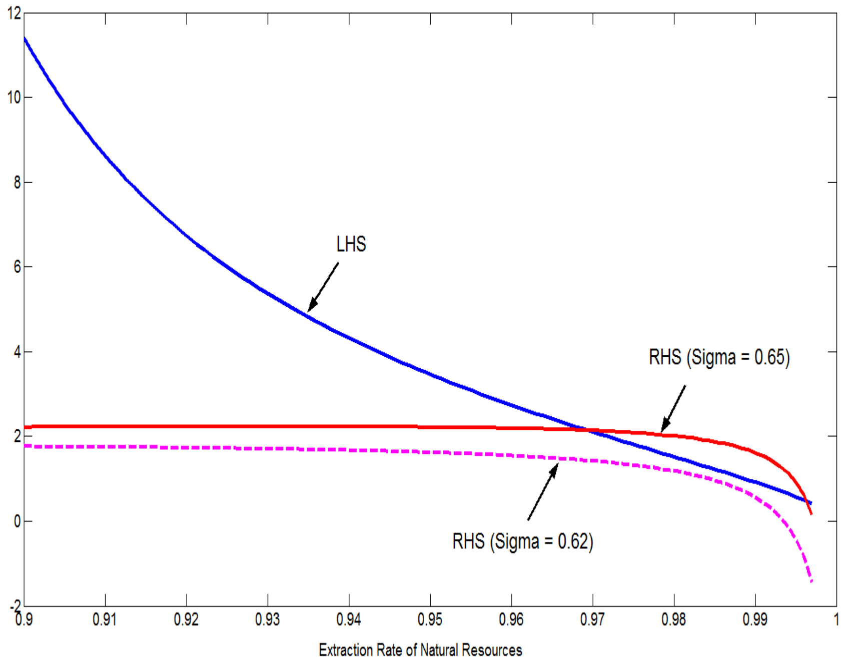

Figure 1 Numerical Example

and $\alpha = 0.24$. Figure 1 plots the left-hand side (LHS) and right-hand side (RHS) of equation (26) under two different values of σ_F, namely 0.62 and 0.65. Both fall within the range of estimates reported by Henningsen *et al.* (2018, Table 4).[17] As shown in this diagram, equation (26) has no solution (which means there is no balanced growth equilibrium) when $\sigma_F = 0.62$ $(\eta = -0.613)$. But when σ_F increases slightly to 0.65 $(\eta = -0.538)$, equation (26) has at least two solutions, which are $\tau^* = 0.9695$ and $\tau^* = 0.9964$. The possibility of multiple equilibria may pose a challenge in deriving general theoretical results, but it does not alter the fundamental nature of the AGI solution — in each of these equilibria, the long-run growth rate γ^* is endogenously determined in the model.

We now turn to the main properties of a balanced growth equilibrium when $\sigma_G(\widehat{x}) \neq 1$ for all $\widehat{x} > 0$.

[17] In Henningsen *et al.* (2018, Table 4), the elasticity of substitution between the inputs of $F(\cdot)$ is denoted by $\sigma_{(LE)K}$. In the existing empirical studies, it is conventional to use commerical energy consumption as a proxy for natural resource input.

Theorem 2 *Suppose Assumptions A1 and A2 are satisfied. Suppose the elasticity of substitution of $G(\cdot)$ is never equal to one. Then any balanced growth equilibrium (if exists) must satisfy $\gamma^* = 1+a$, $r^* = q$, and*

$$1 - \tau^* = \frac{(1+a)(1+n)}{1+q}. \tag{27}$$

In addition, such an equilibrium will have $\widehat{k}_t = \widehat{k}^*$ and $\widehat{x}_t = \widehat{x}^*$ for all t, where $\widehat{k}^*$ and $\widehat{x}^*$ are determined by

$$F_1\left(\widehat{k}^*, G(\widehat{x}^*, 1)\right) = q + \delta, \tag{28}$$

$$(1+a)(1+n)\widehat{k}^* = F_2\left(\widehat{k}^*, G(\widehat{x}^*, 1)\right)\left[\frac{G_2(\widehat{x}^*, 1)}{2+\theta} - \left(\frac{1-\tau^*}{\tau^*}\right)\widehat{x}^* G_1(\widehat{x}^*, 1)\right]. \tag{29}$$

Theorem 2 presents a solution that is in stark contrast to the AGI solution. Specifically, if the elasticity of substitution of $G(\cdot)$ is bounded away from one, then either there is no balanced growth equilibrium or any such equilibrium will have a common growth rate in per-capita variables that is solely determined by the exogenous growth factor A_t. Thus, there is no room for endogenous growth. This theorem also highlights two important differences between the two technological factors A_t and Q_t. First, the growth rate of A_t determines the growth rate of per-capita variables (γ^*), while the growth rate of Q_t determines the rate of return from physical capital (r^*). Second, holding other factors constant, a higher growth rate in A_t will suppress the utilisation rate τ^* while a higher growth rate in Q_t will promote it. Since τ^* must be confined between zero and one, it is necessary to impose the restriction $1 + q > (1+a)(1+n)$. In particular, the growth rate of resource-augmenting technological factor Q_t must remain strictly positive, even when there is no population growth (i.e., $n = 0$) and no labour-augmenting technological progress (i.e., $a = 0$). This shows that a sufficiently high pace of resource-augmenting technological progress is essential for the exogenous growth solution.

To shed some light on the existence and uniqueness of the exogenous growth solution, we focus on the case when both $F(\cdot)$ and $G(\cdot)$ take the CES form in (8) and (9). But unlike Proposition 1, there is no need to impose any restriction on $\sigma_F = (1-\eta)^{-1}$. Define an auxiliary notation Θ according to

$$\Theta \equiv \frac{q+\delta}{\alpha(2+\theta)}\left[\left(\frac{q+\delta}{\alpha}\right)^{\frac{\eta}{1-\eta}} - \alpha\right].$$

Proposition 2 *Suppose* $F(\cdot)$ and $G(\cdot)$ *are given by (8) and (9), respectively. Suppose further that* $\min\{\Theta, 1+q\} > (1+a)(1+n)$. *Then there exists a unique balanced growth equilibrium that satisfies* $\gamma^* = 1+a$, $r^* = q$, *and (27)-(29).*

It is worth mentioning that the above result covers the special case in which $F(\cdot)$ and $G(\cdot)$ have the same constant elasticity of substitution, i.e., $\eta = \psi$. In this case, the production function in (7) becomes

$$Y_t = \left[\alpha K_t^\eta + (1-\alpha)\varphi(Q_t X_t)^\eta + (1-\alpha)(1-\varphi)(A_t N_t)^\eta\right]^{\frac{1}{\eta}},$$

which is the familiar Dixit–Stiglitz aggregator function. It is also worth pointing out the main results of our Theorem 1 and Theorem 2 can be readily extended to an environment with infinitely-lived consumers. The details are shown in Appendix C.

We conclude this section with a heuristic discussion on the results so far. Using the first-order conditions in (11) and (12), we can obtain

$$\frac{Q_t X_t}{A_t N_t} \cdot \frac{G_1(Q_t X_t, A_t N_t)}{G_2(Q_t X_t, A_t N_t)} = \frac{p_t X_t / Y_t}{w_t N_t / Y_t}.$$

On the left side of this equation, we have the ratio between effective resource input $(Q_t X_t)$ and effective labour input $(A_t N_t)$ multiplied by the marginal rate of technical substitution between the two. On the right side is the ratio between the share of aggregate output expended on resource input $(p_t X_t / Y_t)$ and the share distributed as wages $(w_t N_t / Y_t)$. Taking the logarithm of both sides gives

$$\ln\left(\frac{Q_t X_t}{A_t N_t}\right) + \ln\left[\frac{G_1(Q_t X_t, A_t N_t)}{G_2(Q_t X_t, A_t N_t)}\right] = \ln\left(\frac{p_t X_t / Y_t}{w_t N_t / Y_t}\right).$$

In any balanced growth equilibrium, the relative share on the right must remain constant but the variables on the left can be changing over time. In terms of total derivatives, this can be expressed as

$$d\ln\left(\frac{Q_t X_t}{A_t N_t}\right) + d\ln\left[\frac{G_1(Q_t X_t, A_t N_t)}{G_2(Q_t X_t, A_t N_t)}\right] = 0$$

$$\Rightarrow d\ln\left(\frac{Q_t X_t}{A_t N_t}\right)\left\{1 + \frac{d\ln\left[\frac{G_1(Q_t X_t, A_t N_t)}{G_2(Q_t X_t, A_t N_t)}\right]}{d\ln\left(\frac{Q_t X_t}{A_t N_t}\right)}\right\} = \frac{d\widehat{x}_t}{\widehat{x}_t} \cdot \left[1 - \frac{1}{\sigma_G(\widehat{x}_t)}\right] = 0,$$

where $\widehat{x}_t$ and $\sigma_G(\widehat{x}_t)$ are as defined before. The last equality succinctly summarises our main results. Specifically, if $\sigma_G(\cdot)$ is never equal to one, then this condition holds if and only if $\widehat{x}_t$ is constant in any balanced growth equilibrium. This leads to equation (27) in Theorem 2. If $\sigma_G(\cdot)$ is identical to one, then this condition is automatically satisfied and $\widehat{x}_t$ can be changing over time along any balanced growth path. In this case, equation (27) will be scrapped and the utilisation rate τ^* will be determined by other factors.

4 Further Results and Discussions

4.1 Alternative Specifications of Production Function

In this subsection, we will consider two alternative specifications of the production function. These are given by

$$Y_t = F\left(A_t N_t, G\left(K_t, Q_t X_t\right)\right), \tag{30}$$

$$Y_t = F\left(Q_t X_t, G\left(K_t, A_t N_t\right)\right). \tag{31}$$

To preserve consistency across all three specifications, we use $G(\cdot)$ to represent the "inner" aggregator function and $F(\cdot)$ to represent the "outer" aggregator function in (7), (30) and (31). All three specifications will coincide with AGI's production function if both $G(\cdot)$ and $F(\cdot)$ have the Cobb-Douglas form. Our main interest here is to examine the properties of balanced growth equilibrium when *one* of the aggregator functions in (30) and (31) *does not* take the Cobb-Douglas form. To this end, we consider four different parametric production functions based on (30) and (31). In the first two specifications, the inner aggregator function is Cobb-Douglas but the outer one has a CES form, so that

$$Y_t = \left\{\varphi\left(A_t N_t\right)^{\psi} + (1-\varphi)\left[K_t^{\alpha}\left(Q_t X_t\right)^{1-\alpha}\right]^{\psi}\right\}^{\frac{1}{\psi}}, \tag{32}$$

$$Y_t = \left\{\varphi\left(Q_t X_t\right)^{\psi} + (1-\varphi)\left[K_t^{\alpha}\left(A_t N_t\right)^{1-\alpha}\right]^{\psi}\right\}^{\frac{1}{\psi}}, \tag{33}$$

with $\alpha \in (0,1)$, $\varphi \in (0,1)$ and $\psi < 1$. In the second group, the inner aggregator function is a CES function and the outer one is Cobb-Douglas, so that

$$Y_t = \left[\varphi K_t^{\psi} + (1-\varphi)\left(Q_t X_t\right)^{\psi}\right]^{\frac{1-\beta}{\psi}}\left(A_t N_t\right)^{\beta} \tag{34}$$

$$Y_t = \left(Q_t X_t\right)^{v}\left[\varphi K_t^{\psi} + (1-\varphi)\left(A_t N_t\right)^{\psi}\right]^{\frac{1-v}{\psi}}, \tag{35}$$

with $\beta \in (0,1)$, $v \in (0,1)$, $\varphi \in (0,1)$ and $\psi < 1$.[18] The main result of this subsection is summarised in Theorem 3.

Theorem 3 *Suppose the production function takes one of the forms in (32)-(35). Then any balanced growth equilibrium (if exists) must satisfy $\gamma^* = 1 + a$, $r^* = q$, and*

$$1 - \tau^* = \frac{(1+a)(1+n)}{1+q}.$$

The main message of Theorem 3 is clear: despite the differences in appearance, all the production functions in (32)-(35) have the same implications for balanced growth equilibrium. Specifically, any balanced growth equilibrium (if exists) must satisfy $\gamma^* = 1 + a$, $r^* = q$, and $(1 - \tau^*) = (1+a)(1+n)/(1+q)$. It follows that the two transformed variables $\widehat{k}_t$ and $\widehat{x}_t$ must be time-invariant in this type of equilibrium, and hence there is no room for endogenous economic growth.

4.2 Discussions

The results in the previous sections suggest that the AGI solution is valid only when the elasticity of substitution between labour and natural resources is constant and equal to one. If we rewrite (19) as

$$G(Q_tX_t, A_tN_t) = \left[A_t (Q_tX_t)^{\frac{1-\phi}{\phi}} N_t\right]^{\phi},$$

the the expression $\widetilde{X}_t \equiv A_t (Q_tX_t)^{\frac{1-\phi}{\phi}}$ can be viewed as a labour-augmenting factor and serves as the engine of growth. When viewed through this lens, our result suggests that the AGI solution is valid only when effective resource input is labour-augmenting in the production function, i.e.,

$$Y_t = F\left(K_t, \left(\widetilde{X}_tN_t\right)^{\phi}\right).$$

This result may look similar to the celebrated Uzawa Growth Theorem [Uzawa (1961)]. But there are at least two important differences between the two. First, the Uzawa Growth Theorem and its variants are typically derived from a CRTS production function with only two inputs, namely physical capital and labour [see, for instance, Uzawa (1961), Schlicht (2006), Jones and Scrimgeour (2008) and Grossman *et al.* (2017)]. It is not immediately clear how the Uzawa Theorem can be extended to a general CRTS production function with more than two inputs, such as the one

[18] The parameters β and v have the same economic meaning as in AGI. Specifically, they represent the share of total output distributed as labour income and expenses on natural resource input.

considered in the current study. Second, and more importantly, the Uzawa Growth Theorem states the conditions under which a balanced growth equilibrium can emerge, without explicitly mentioning whether the "engine of growth" is exogenous or endogenous. The distinction between exogenous and endogenous growth, however, is at the centre of our analysis. In particular, our results are intended to clarify the conditions under which endogenous economic growth can emerge in the neoclassical growth model with non-renewable resources.

Whether the elasticity of substitution between labour input and resource input is equal to one is ultimately an empirical question. A number of existing studies have provided estimates on the elasticity of substitution between physical capital, labour and commercial energy consumption.[19] The last one is typically viewed as a proxy for natural resource input. These studies usually report a less-than-unity elasticity of substitution between labour and energy [Kemfert (1998), Kemfert and Welsch (2000) and van der Werf (2008)], thus casting doubt on the empirical relevance of the AGI solution.

5 Conclusions

In this paper, we re-examine the possibility of endogenous long-term economic growth in neoclassical models with non-renewable resources. Unlike most of the existing studies which focus exclusively on Cobb-Douglas production function, we adopt a general specification of production technology and seek general conditions under which endogenous economic growth can emerge. Our results suggest that this can happen only when the elasticity of substitution between labour and natural resources is constant and equal to one. This condition, however, has found little support in empirical studies. For all other specifications that we have considered, including those that are supported by empirical evidence, the model predicts that long-term economic growth is entirely driven by the exogenous labour-augmenting technological factor. This has the stark implication that practices and policies related to natural resource utilisation and management are irrelevant to long-term economic growth. Our results thus expose the difficulties of using the standard one-sector neoclassical model to analyse the relationship between natural resources and economic growth. A multi-sector model, or one that explicitly accounts for productive government spending and R&D activities, is perhaps more suitable for this line of research.

[19] See van der Werf (2008) and Henningsen *et al.* (2018) for literature review and discussions on different estimation strategies.

Appendix A: Nested CES Production Functions

In this appendix, we will verify that Assumption A2 is satisfied by all the nested CES production functions considered in Sections 3 and 4. We begin with the specification considered in Section 3, which is

$$F(K_t, Z_t) = \left[\alpha K_t^{\eta} + (1-\alpha) Z_t^{\eta}\right]^{\frac{1}{\eta}}, \qquad \text{with } \alpha \in (0,1) \text{ and } \eta < 1,$$

$$G(Q_t X_t, A_t N_t) \equiv \left[\varphi (Q_t X_t)^{\psi} + (1-\varphi)(A_t N_t)^{\psi}\right]^{\frac{1}{\psi}}, \qquad \text{with } \varphi \in (0,1) \text{ and } \psi < 1.$$

First, consider capital input. If $\eta \leq 0$, then

$$\lim_{K_t \to 0} F(K_t, G(Q_t X_t, A_t N_t)) = 0$$

regardless of the value of ψ. In other words, physical capital is essential for production when $\eta \leq 0$. If $\eta \in (0,1)$, then

$$\lim_{K_t \to 0} F_1(K_t, G(Q_t X_t, A_t N_t)) = \infty,$$

regardless of the value of ψ. Next, consider the inputs of $G(\cdot)$. When $\psi \leq 0$, we have

$$\lim_{X_t \to 0} G(Q_t X_t, A_t N_t) = \lim_{N_t \to 0} G(Q_t X_t, A_t N_t) = 0,$$

$$\lim_{X_t \to 0} G_1(Q_t X_t, A_t N_t) = \varphi^{\frac{1}{\psi}} Q_t \quad \text{and} \quad \lim_{N_t \to 0} G_2(Q_t X_t, A_t N_t) = (1-\varphi)^{\frac{1}{\psi}} A_t.$$

There are now two subcases to consider: If $\psi \leq 0$ and $\eta \leq 0$, then both natural resources and labour are essential for production. In particular, we can show that

$$\lim_{X_t \to 0} F(K_t, G(Q_t X_t, A_t N_t)) = \lim_{N_t \to 0} F(K_t, G(Q_t X_t, A_t N_t)) = 0.$$

If $\psi \leq 0$ and $\eta \in (0,1)$, then we can show that

$$\lim_{X_t \to 0} \frac{\partial Y_t}{\partial X_t} = (1-\alpha) \left\{ \alpha \lim_{X_t \to 0} \left[\frac{G(Q_t X_t, A_t N_t)}{K_t} \right]^{-\eta} + 1 - \alpha \right\}^{\frac{1}{\eta} - 1} \cdot \lim_{X_t \to 0} G_1(Q_t X_t, A_t N_t),$$

$$\lim_{N_t \to 0} \frac{\partial Y_t}{\partial N_t} = (1-\alpha) \left\{ \alpha \lim_{N_t \to 0} \left[\frac{G(Q_t X_t, A_t N_t)}{K_t} \right]^{-\eta} + 1 - \alpha \right\}^{\frac{1}{\eta} - 1} \cdot \lim_{N_t \to 0} G_2(Q_t X_t, A_t N_t).$$

Both of these limits diverge to infinity as

$$\lim_{X_t \to 0} \left[\frac{G(Q_t X_t, A_t N_t)}{K_t} \right]^{-\eta} = \lim_{N_t \to 0} \left[\frac{G(Q_t X_t, A_t N_t)}{K_t} \right]^{-\eta} = \infty.$$

If $\psi \in (0,1)$, then we have

$$\lim_{X_t \to 0} G(Q_t X_t, A_t N_t) = (1-\varphi)^{\frac{1}{\psi}} (A_t N_t) \qquad \text{and} \qquad \lim_{N_t \to 0} G(Q_t X_t, A_t N_t) = \varphi^{\frac{1}{\psi}} (Q_t X_t),$$

$$\lim_{X_t \to 0} G_1(Q_t X_t, A_t N_t) = \lim_{N_t \to 0} G_2(Q_t X_t, A_t N_t) = \infty.$$

Using these we can obtain

$$\lim_{X_t \to 0} \frac{\partial Y_t}{\partial X_t} = F_2\left(K_t, (1-\varphi)^{\frac{1}{\psi}} A_t N_t\right) \left[\lim_{X_t \to 0} G_1(Q_t X_t, A_t N_t)\right] = \infty,$$

$$\lim_{N_t \to 0} \frac{\partial Y_t}{\partial N_t} = F_2\left(K_t, \varphi^{\frac{1}{\psi}} Q_t X_t\right) \left[\lim_{N_t \to 0} G_2(Q_t X_t, A_t N_t)\right] = \infty.$$

Note that these results hold regardless of the value of η.

Next, we turn to the production function in (32). There are now only two possible cases: If $\psi \leq 0$, then all three inputs are essential for production. If $\psi \in (0,1)$, then we can obtain

$$\lim_{N_t \to 0} \frac{\partial Y_t}{\partial N_t} = \varphi A_t \left\{ \varphi + (1-\varphi) \lim_{N_t \to 0} \left[\frac{A_t N_t}{K_t^{\alpha} (Q_t X_t)^{1-\alpha}} \right]^{-\psi} \right\}^{\frac{1}{\psi}-1} = \infty,$$

$$\lim_{K_t \to 0} \frac{\partial Y_t}{\partial K_t} = \alpha(1-\varphi) \left\{ \varphi \lim_{N_t \to 0} \left[\frac{K_t^{\alpha} (Q_t X_t)^{1-\alpha}}{A_t N_t} \right]^{-\psi} + 1 - \varphi \right\}^{\frac{1}{\psi}-1} \left[\lim_{N_t \to 0} \left(\frac{K_t}{Q_t X_t} \right)^{\alpha-1} \right] = \infty,$$

$$\begin{aligned} \lim_{X_t \to 0} \frac{\partial Y_t}{\partial X_t} &= (1-\alpha)(1-\varphi) \left\{ \varphi \lim_{X_t \to 0} \left[\frac{K_t^{\alpha} (Q_t X_t)^{1-\alpha}}{A_t N_t} \right]^{-\psi} + 1 - \varphi \right\}^{\frac{1}{\psi}-1} \left[\lim_{X_t \to 0} \left(\frac{K_t}{Q_t X_t} \right)^{\alpha} \right] \\ &= \infty. \end{aligned}$$

Note that the production functions in (32) and (33) are essentially identical, except that $A_t N_t$ and $Q_t X_t$ have switched place. Thus, using the same line of argument we can show that (33) satisfies Assumption A2.

We now consider the production function in (34). The first thing to note is that labour input is essential for production regardless of the value of ψ. If $\psi \leq 0$, then both physical capital and natural resources are essential for production. What remains is to consider the marginal product of these inputs when $\psi \in (0,1)$. Straightforward differentiation gives

$$\frac{\partial Y_t}{\partial K_t} = (1-\beta)\,\varphi\,(A_t N_t)^{\beta}\left[\varphi + (1-\varphi)\left(\frac{K_t}{Q_t X_t}\right)^{-\psi}\right]^{\frac{1}{\psi}-1}\left[\varphi K_t^{\psi} + (1-\varphi)\,(Q_t X_t)^{\psi}\right]^{-\frac{\beta}{\psi}},$$

$$\frac{\partial Y_t}{\partial X_t} = (1-\beta)\,(1-\varphi)\,(A_t N_t)^{\beta}\left[\varphi\left(\frac{Q_t X_t}{K_t}\right)^{-\psi} + (1-\varphi)\right]^{\frac{1}{\psi}-1}\left[\varphi K_t^{\psi} + (1-\varphi)\,(Q_t X_t)^{\psi}\right]^{-\frac{\beta}{\psi}}.$$

Since

$$\lim_{K_t \to 0}\left[\varphi + (1-\varphi)\left(\frac{K_t}{Q_t X_t}\right)^{-\psi}\right]^{\frac{1}{\psi}-1} = \lim_{X_t \to 0}\left[\varphi\left(\frac{Q_t X_t}{K_t}\right)^{-\psi} + (1-\varphi)\right]^{\frac{1}{\psi}-1} = \infty,$$

it follows that

$$\lim_{K_t \to 0}\frac{\partial Y_t}{\partial K_t} = (1-\beta)\,\varphi\,(1-\varphi)^{-\frac{\beta}{\psi}}\left(\frac{Q_t X_t}{A_t N_t}\right)^{-\beta}\lim_{K_t \to 0}\left[\varphi + (1-\varphi)\left(\frac{K_t}{Q_t X_t}\right)^{-\psi}\right]^{\frac{1}{\psi}-1} = \infty,$$

$$\lim_{X_t \to 0}\frac{\partial Y_t}{\partial X_t} = (1-\beta)\,\varphi^{-\frac{\beta}{\psi}}\,(1-\varphi)\left(\frac{K_t}{A_t N_t}\right)^{-\beta}\lim_{X_t \to 0}\left[\varphi\left(\frac{Q_t X_t}{K_t}\right)^{-\psi} + (1-\varphi)\right]^{\frac{1}{\psi}-1} = \infty.$$

Since (34) and (35) are symmetric, the same line of argument can be used to show the desired properties for (35).

Appendix B: Proofs

Proof of Theorem 1

The proof is divided into a number of steps:

Step 1 This part of the proof uses the same line of argument as in Schlicht (2006) and Jones and Scrimgeour (2008). In any balanced growth equilibrium, Y_t grows at a constant rate $\widehat{\gamma} \equiv \gamma^* (1+n)$ in every period, so that $Y_{t+1} = \widehat{\gamma} Y_t$, for all t. Rearranging terms and applying the CRTS property of $F(\cdot)$ gives

$$\begin{aligned} Y_t &= F\left(\widehat{\gamma}^{-1} K_{t+1}, \widehat{\gamma}^{-1} G(Q_{t+1}X_{t+1}, A_{t+1}N_{t+1})\right) \\ &= F\left(K_t, \widehat{\gamma}^{-1} G(Q_{t+1}X_{t+1}, A_{t+1}N_{t+1})\right). \end{aligned}$$

The second line uses the condition that K_t and Y_t grow at the same rate in any balanced growth equilibrium. For any $K_t > 0$, $F(K_t, \cdot)$ is a strictly increasing function. Hence, the following equality must be satisfied in any balanced growth equilibrium,

$$G(Q_t X_t, A_t N_t) = \widehat{\gamma}^{-1} G(Q_{t+1}X_{t+1}, A_{t+1}N_{t+1}). \tag{36}$$

Note that (36) holds regardless of whether $G(\cdot)$ is Cobb-Douglas.

Suppose now $G(\cdot)$ is given by

$$G(Q_t X_t, A_t N_t) = (Q_t X_t)^{1-\phi} (A_t N_t)^{\phi}, \qquad \text{for some } \phi \in (0,1).$$

Combining this with $A_{t+1} = (1+a) A_t$, $Q_{t+1} = (1+q) Q_t$, $X_{t+1} = (1-\tau^*) X_t$ and $N_{t+1} = (1+n) N_t$, we can rewrite (36) as

$$(Q_t X_t)^{1-\phi} (A_t N_t)^{\phi} = \widehat{\gamma}^{-1} [(1+q)(1-\tau^*)] [(1+a)(1+n)]^{\phi} (Q_t X_t)^{1-\phi 1-\phi} (A_t N_t)^{\phi}.$$

If we ignore the trivial case in which $(Q_t X_t)^{1-\phi} (A_t N_t)^{\phi} = 0$, then (36) is valid if and only if

$$[(1+q)(1-\tau^*)]^{1-\phi} [(1+a)(1+n)]^{\phi} = \widehat{\gamma} \equiv \gamma^* (1+n)$$

$$\Rightarrow \gamma^* = (1+a)^{\phi} \left[\frac{(1+q)(1-\tau^*)}{1+n}\right]^{1-\phi}.$$

This is equation (20) in the theorem.

Step 2 Next, we will show that in any balanced growth equilibrium with a constant rate of return $r^* > -\delta$, the ratio $p_t X_t / Y_t$ must be time-invariant and strictly positive. This can then be used to derive equation (21). Suppose $r_t = r^* > -\delta$. Then by (10), we have

$$F_1\left(1, \frac{G(Q_t X_t, A_t N_t)}{K_t}\right) = F_1\left(1, \frac{\widehat{x}_t^{1-\phi}}{\widehat{k}_t}\right) = r^* + \delta > 0.$$

Since $F_1(1, \cdot)$ is strictly decreasing, it follows that the ratio between $\widehat{x}_t^{1-\phi}$ and $\widehat{k}_t$ must be constant in any balanced growth equilibrium. Hence, we can write

$$\frac{G(Q_t X_t, A_t N_t)}{K_t} = \frac{\widehat{x}_t^{1-\phi}}{\widehat{k}_t} = \chi^* > 0. \tag{37}$$

By the homogeneity properties of $F(\cdot)$ and $F_2(\cdot)$, we can write

$$F_2(K_t, G(Q_t X_t, A_t N_t)) = F_2(1, \chi^*),$$

$$F(K_t, G(Q_t X_t, A_t N_t)) = K_t F(1, \chi^*).$$

Using these and (11), we can get

$$\begin{aligned}
\frac{p_t X_t}{Y_t} &= \frac{Q_t X_t}{K_t} \frac{F_2(1, \chi^*) G_1(Q_t X_t, A_t N_t)}{F(1, \chi^*)} \\
&= \frac{F_2(1, \chi^*)}{F(1, \chi^*)} \frac{G(Q_t X_t, A_t N_t)}{K_t} \frac{Q_t X_t G_1(Q_t X_t, A_t N_t)}{G(Q_t X_t, A_t N_t)} \\
&= (1-\phi) \frac{\chi^* F_2(1, \chi^*)}{F(1, \chi^*)}.
\end{aligned}$$

Hence, $p_t X_t / Y_t$ must be strictly positive and time-invariant. This in turn implies

$$\frac{p_{t+1}}{p_t} \frac{X_{t+1}}{X_t} = (1 + r^*)(1 - \tau^*) = \frac{Y_{t+1}}{Y_t} = \gamma^*(1+n).$$

Step 3 We now derive equation (22), which is based on the capital market clearing condition. In any competitive equilibrium, the market for physical capital clears when

$$K_{t+1} = N_t s_t = N_t \left(\frac{w_t}{2+\theta} - p_t m_t\right).$$

The second equality follows from (6). Substituting (12) and (11) into the above equation gives

$$K_{t+1} = F_2\left(K_t, G\left(Q_tX_t, A_tN_t\right)\right)\left[\frac{1}{2+\theta}A_tN_tG_2\left(Q_tX_t, A_tN_t\right) - N_tm_tQ_tG_1\left(Q_tX_t, A_tN_t\right)\right]. \quad (38)$$

As shown in Step 2, we can rewrite $F_2\left(K_t, G\left(Q_tX_t, A_tN_t\right)\right)$ as $F_2\left(1, \chi^*\right)$. Using the market clearing condition for natural resources, we can get

$$N_tm_t = M_{t+1} = \left(1-\tau^*\right)\frac{M_t}{X_t}\cdot X_t = \left(\frac{1-\tau^*}{\tau^*}\right)X_t.$$

Substituting these into (38) gives

$$K_{t+1} = F_2\left(1, \chi^*\right)\left[\frac{1}{2+\theta}A_tN_tG_2\left(Q_tX_t, A_tN_t\right) - \left(\frac{1-\tau^*}{\tau^*}\right)Q_tX_tG_1\left(Q_tX_t, A_tN_t\right)\right].$$

Finally, using the Cobb-Douglas specification for $G(\cdot)$, we can simplify this to become

$$K_{t+1} = F_2\left(1, \chi^*\right)\left[\frac{\phi}{2+\theta} - \left(\frac{1-\tau^*}{\tau^*}\right)(1-\phi)\right]G\left(Q_tX_t, A_tN_t\right).$$

Dividing both sides by K_t and using (37) once more gives

$$\frac{K_{t+1}}{K_t} = \gamma^*(1+n) = \chi^*F_2\left(1, \chi^*\right)\left[\frac{\phi}{2+\theta} - \left(\frac{1-\tau^*}{\tau^*}\right)(1-\phi)\right].$$

This completes the proof of Theorem 1.

Proof of Proposition 1

Using (20) and (21), we can get

$$\gamma^*(1+n) = (1+b)(1+n)^{\phi}\left(1-\tau^*\right)^{1-\phi},$$

$$r^* = (1+b)(1+n)^{\phi}\left(1-\tau^*\right)^{-\phi} - 1 \equiv r\left(\tau^*\right).$$

Using (8), we can derive

$$F_1\left(1, \chi^*\right) = \alpha\left[\alpha + (1-\alpha)\left(\chi^*\right)^{\eta}\right]^{\frac{1-\eta}{\eta}},$$

$$F_2\left(1, \chi^*\right) = (1-\alpha)\left(\chi^*\right)^{\eta-1}\left[\alpha + (1-\alpha)\left(\chi^*\right)^{\eta}\right]^{\frac{1-\eta}{\eta}}.$$

Equation (23) then implies

$$(1-\alpha)(\chi^*)^\eta = \left[\frac{r(\tau^*)+\delta}{\alpha}\right]^{\frac{\eta}{1-\eta}} - \alpha.$$

It follows that

$$\begin{aligned}\chi^* F_2(1,\chi^*) &= (1-\alpha)(\chi^*)^\eta \left[\alpha + (1-\alpha)(\chi^*)^\eta\right]^{\frac{1-\eta}{\eta}} \\ &= \frac{r(\tau^*)+\delta}{\alpha}\left\{\left[\frac{r(\tau^*)+\delta}{\alpha}\right]^{\frac{\eta}{1-\eta}} - \alpha\right\}.\end{aligned}$$

Using these expressions, we can rewrite (22) as

$$\frac{(2+\theta)(1+b)(1+n)^\phi (1-\tau^*)^{1-\phi}}{\phi - \left(\frac{1-\tau^*}{\tau^*}\right)(2+\theta)(1-\phi)} = \frac{r(\tau^*)+\delta}{\alpha}\left\{\left[\frac{r(\tau^*)+\delta}{\alpha}\right]^{\frac{\eta}{1-\eta}} - \alpha\right\}.$$

A unique balanced growth equilibrium exists if there is a unique solution for this equation. Define two auxiliary functions $\Lambda(\cdot)$ and $\Gamma(\cdot)$ according to

$$\Lambda(\tau) \equiv \frac{(2+\theta)(1+b)(1+n)^\phi (1-\tau)^{1-\phi}}{\phi - \left(\frac{1-\tau}{\tau}\right)(2+\theta)(1-\phi)},$$

$$\Gamma(\tau) \equiv \frac{r(\tau)+\delta}{\alpha}\left\{\left[\frac{r(\tau)+\delta}{\alpha}\right]^{\frac{\eta}{1-\eta}} - \alpha\right\}.$$

The following properties of $\Lambda(\cdot)$ can be easily verified: $\Lambda(1) = 0$; $\Lambda(\tau) \to \infty$ as $\tau \to \overline{\tau}$, where $\overline{\tau} \in (0,1)$ is the threshold value defined in (24); $\Lambda(\tau) < 0$ for all $\tau < \overline{\tau}$; and $\Lambda(\cdot)$ is strictly decreasing over the range $(\overline{\tau}, 1]$. Similarly, one can show that $\Gamma(\overline{\tau}) < \infty$ and $\Gamma(\tau) \to \infty$ as $\tau \to 1$ if $\eta \in (0,1)$. Since both $\Lambda(\cdot)$ and $\Gamma(\cdot)$ are continuous over $(\overline{\tau}, 1]$, these properties ensure the existence of at least one value $\tau^* \in (\overline{\tau}, 1)$ such that $\Lambda(\tau^*) = \Gamma(\tau^*)$.

If, in addition, $\Gamma(\cdot)$ is strictly increasing over $(\overline{\tau}, 1]$, then a unique solution exists. Straightforward differentiation shows that

$$\Gamma'(\tau) = \frac{1}{\alpha}\left\{\frac{1}{1-\eta}\left[\frac{r(\tau)+\delta}{\alpha}\right]^{\frac{\eta}{1-\eta}} - \alpha\right\}(1+b)(1+n)^\phi \phi (1-\tau)^{-(1+\phi)}.$$

Hence, $\Gamma'(\tau) \gtrless 0$ if and only if

$$\left[\frac{r(\tau)+\delta}{\alpha}\right]^{\frac{\eta}{1-\eta}} \gtrless \alpha(1-\eta) \Leftrightarrow r(\tau)+\delta \gtrless \alpha^{\frac{1}{\eta}}(1-\eta)^{\frac{1}{\eta}-1}.$$

Since $r(\tau)$ is a strictly increasing function, it follows that $\Gamma(\cdot)$ is strictly increasing over $(\overline{\tau}, 1]$ if

$$r(\overline{\tau}) + \delta > \alpha^{\frac{1}{\eta}} (1-\eta)^{\frac{1}{\eta}-1}.$$

This condition can be rewritten as (25). This completes the proof of Proposition 1.

Proof of Theorem 2

Step 1 First, we will show that $\gamma^* = 1 + a$ if the elasticity of substitution of $G(\cdot)$ is never equal to one. Recall that equation (36) in the proof of Theorem 1 is valid even if $G(\cdot)$ is not Cobb-Douglas. Define $\widehat{x}_t \equiv Q_t X_t / (A_t N_t)$. Then by the CRTS property of $G(\cdot)$, equation (36) can be equivalently stated as

$$G(Q_t X_t, A_t N_t) = G\left[\frac{(1+q)(1-\tau^*)}{\widehat{\gamma}} Q_t X_t, \frac{(1+a)(1+n)}{\widehat{\gamma}} A_t N_t,\right].$$

Define the following notations

$$\varsigma \equiv \frac{(1+a)(1+n)}{\widehat{\gamma}} \qquad \text{and} \qquad \eta \equiv \frac{(1+q)(1-\tau^*)}{\widehat{\gamma}}.$$

Dividing both sides by $\varsigma A_t N_t$ and using $g(\widehat{x}) \equiv G(\widehat{x}, 1)$ give

$$g(\widehat{x}_t) = \varsigma g\left(\frac{\eta}{\varsigma}\widehat{x}_t\right), \qquad \text{for all } \widehat{x}_t > 0. \tag{39}$$

Equation (39) is trivially satisfied if $\varsigma = \eta = 1$, which immediately implies

$$\gamma^* = 1 + a \qquad \text{and} \qquad 1 - \tau^* = \frac{(1+a)(1+n)}{1+q}.$$

We now show that if $\sigma_G(\cdot) \neq 1$, then equation (39) holds if and only if $\varsigma = \eta = 1$.

We first establish an intermediate result: For any $\widehat{x} > 0$,

$$\frac{d}{d\widehat{x}}\left[\frac{\widehat{x} g'(\widehat{x})}{g(\widehat{x})}\right] \gtrless 0 \qquad \text{if and only if} \qquad \sigma_G(\widehat{x}) \gtrless 1.$$

To start, straightforward differentiation gives

$$\frac{d}{d\widehat{x}}\left[\frac{\widehat{x} g'(\widehat{x})}{g(\widehat{x})}\right] = \frac{g'(\widehat{x})}{g(\widehat{x})} - \frac{\widehat{x}[g'(\widehat{x})]^2}{[g(\widehat{x})]^2} + \frac{\widehat{x} g''(\widehat{x})}{g(\widehat{x})}. \tag{40}$$

Next, using the expression in (18), $\sigma_G(\widehat{x}) \gtreqless 1$ if and only if

$$\frac{g'(\widehat{x})\left[g(\widehat{x}) - \widehat{x}g'(\widehat{x})\right]}{g(\widehat{x})} \gtreqless -\widehat{x}g''(\widehat{x})$$

$$\Leftrightarrow \frac{g'(\widehat{x})}{g(\widehat{x})}\left[1 - \frac{\widehat{x}g'(\widehat{x})}{g(\widehat{x})}\right] \gtreqless \frac{-\widehat{x}g''(\widehat{x})}{g(\widehat{x})}$$

$$\Leftrightarrow \frac{g'(\widehat{x})}{g(\widehat{x})} - \frac{\widehat{x}\left[g'(\widehat{x})\right]^2}{\left[g(\widehat{x})\right]^2} - \frac{\widehat{x}g''(\widehat{x})}{g(\widehat{x})} = \frac{d}{d\widehat{x}}\left[\frac{\widehat{x}g'(\widehat{x})}{g(\widehat{x})}\right] \gtreqless 0. \tag{41}$$

This intermediate result says that if $\sigma_G(\cdot)$ is never equal to one, then $\widehat{x}g'(\widehat{x})/g(\widehat{x})$ must be either strictly increasing or strictly decreasing for all $\widehat{x} > 0$. We will now apply this result on (39).

Since $g(\cdot)$ is continuously differentiable and (39) holds for all $\widehat{x}_t > 0$, we can differentiate both sides of (39) with respect to $\widehat{x}_t$ and get

$$g'(\widehat{x}_t) = \eta g'\left(\frac{\eta}{\varsigma}\widehat{x}_t\right).$$

Combining this and (39) gives

$$\frac{\widehat{x}_t g'(\widehat{x}_t)}{g(\widehat{x}_t)} = \frac{\frac{\eta}{\varsigma}\widehat{x}_t g'\left(\frac{\eta}{\varsigma}\widehat{x}_t\right)}{g\left(\frac{\eta}{\varsigma}\widehat{x}_t\right)}. \tag{42}$$

As mentioned above, if $\sigma_G(\cdot)$ is never equal to one, then $\widehat{x}g'(\widehat{x})/g(\widehat{x})$ must be either strictly increasing or strictly decreasing for all $\widehat{x} > 0$. Hence, the equality in (42) holds if and only if $\eta = \varsigma$. Using this, we can rewrite (39) as $g'(\widehat{x}_t) = \eta g'(\widehat{x}_t)$, which implies that $\eta = 1$.

Step 2 The equalities $\varsigma = \eta = 1$ in turn imply that $\widehat{k}_t$ and $\widehat{x}_t$ are time-invariant in any balanced growth equilibrium, i.e., $\widehat{k}_t = \widehat{k}^*$ and $\widehat{x}_t = \widehat{x}^*$. Using these, we can rewrite (10) and (11) as

$$r^* + \delta = F_1\left(\widehat{k}^*, G(\widehat{x}^*, 1)\right)$$

$$p_t = Q_t F_2\left(\widehat{k}^*, G(\widehat{x}^*, 1)\right) G_1(\widehat{x}^*, 1).$$

Hence, equation (4) can be used to obtain $r^* = q$. Equation (28) then follows.

Step 3 Finally, we provide the derivation of (29). As shown in the proof of Theorem 1, the capital market clearing condition can be expressed as

$$K_{t+1} = F_2(K_t, G(Q_t X_t, A_t N_t))\left[\frac{1}{2+\theta}A_t N_t G_2(Q_t X_t, A_t N_t) - \left(\frac{1-\tau^*}{\tau^*}\right) Q_t X_t G_1(Q_t X_t, A_t N_t)\right].$$

Dividing both sides by $A_t N_t$ gives

$$(1+a)(1+n)\widehat{k}_{t+1} = F_2\left(\widehat{k}_t, G(\widehat{x}_t, 1)\right)\left[\frac{1}{2+\theta}G_2(\widehat{x}_t, 1) - \left(\frac{1-\tau^*}{\tau^*}\right)\widehat{x}_t G_1(\widehat{x}_t, 1)\right].$$

Equation (29) can be obtained by setting $\widehat{k}_{t+1} = \widehat{k}_t = \widehat{k}^*$ and $\widehat{x}_t = \widehat{x}^*$. This completes the proof of Theorem 2.

Proof of Proposition 2

Suppose $F(\cdot)$ takes the CES form in (8), with $\alpha \in (0,1)$ and $\eta < 1$. Then (28) can be rewritten as

$$\alpha\left\{\alpha + (1-\alpha)\left[\frac{G(\widehat{x}^*, 1)}{\widehat{k}^*}\right]^{\eta}\right\}^{\frac{1-\eta}{\eta}} = q + \delta$$

$$\Rightarrow (1-\alpha)\left[\frac{G(\widehat{x}^*, 1)}{\widehat{k}^*}\right]^{\eta} = \left(\frac{q+\delta}{\alpha}\right)^{\frac{\eta}{1-\eta}} - \alpha$$

Using these, we can write

$$\frac{G(\widehat{x}^*, 1)}{\widehat{k}^*}F_2\left(\widehat{k}^*, G(\widehat{x}^*, 1)\right) = \frac{q+\delta}{\alpha}\left[\left(\frac{q+\delta}{\alpha}\right)^{\frac{\eta}{1-\eta}} - \alpha\right] = (2+\theta)\Theta$$

Similarly, if $G(\cdot)$ takes the CES form in (9), then we can get

$$G_2(\widehat{x}^*, 1) = (1-\varphi)\left[\varphi(\widehat{x}^*)^{\psi} + 1 - \varphi\right]^{\frac{1}{\psi}-1} = \frac{(1-\varphi)G(\widehat{x}^*, 1)}{\varphi(\widehat{x}^*)^{\psi} + 1 - \varphi}$$

$$G_1(\widehat{x}^*, 1) = \frac{\varphi(\widehat{x}^*)^{\psi-1}G(\widehat{x}^*, 1)}{\varphi(\widehat{x}^*)^{\psi} + 1 - \varphi}.$$

Based on these observations, we can rewrite (29) as

$$\begin{aligned}(1+a)(1+n)\left[\varphi(\widehat{x}^*)^{\psi} + 1 - \varphi\right] &= \frac{G(\widehat{x}^*, 1)}{\widehat{k}^*}F_2\left(\widehat{k}^*, G(\widehat{x}^*, 1)\right)\left[\frac{1-\varphi}{2+\theta} - \left(\frac{1-\tau^*}{\tau^*}\right)\varphi(\widehat{x}^*)^{\psi}\right] \\ &= \Theta\left[1 - \varphi - \left(\frac{1-\tau^*}{\tau^*}\right)(2+\theta)\varphi(\widehat{x}^*)^{\psi}\right],\end{aligned}$$

which can be simplified to become

$$(\widehat{x}^*)^{\psi} = \frac{1-\varphi}{\varphi}\frac{\Theta - (1+a)(1+n)}{(1+a)(1+n) + \left(\frac{1-\tau^*}{\tau^*}\right)(2+\theta)\Theta}.$$

The purpose of the additional condition $\min\{\Theta, 1+q\} > (1+a)(1+n)$ is twofold: Firstly, it ensures that a unique, strictly positive value of $\widehat{x}^*$ can be obtained from the above equation. Secondly, it ensures that $\tau^* \in (0,1)$. This completes the proof of Proposition 2.

Proof of Theorem 3

We will consider each of the specifications in (32)-(35) separately.

Specification 1 We begin with the production function in (32). Under this specification, the first-order conditions for the representative firm's problem are given by

$$(1-\varphi)\,\alpha Y_t^{1-\psi} K_t^{\alpha\psi-1} (Q_t X_t)^{(1-\alpha)\psi} = r_t + \delta, \tag{43}$$

$$(1-\varphi)(1-\alpha)\,Y_t^{1-\psi} K_t^{\alpha\psi} (Q_t X_t)^{(1-\alpha)\psi-1} Q_t = p_t, \tag{44}$$

$$\varphi Y_t^{1-\psi} (A_t N_t)^{\psi-1} A_t = w_t. \tag{45}$$

In any balanced growth equilibrium, both the capital-output ratio and interest rate are constant over time, i.e.,

$$Y_t = \frac{1}{\kappa^*} K_t, \qquad \text{for some } \kappa^* > 0,$$

and $r_t = r^*$, for some $r^* > -\delta$. Substituting these into (43) gives

$$(1-\varphi)\,\alpha\,(\kappa^*)^{\psi-1} \left(\frac{K_t}{Q_t X_t}\right)^{(\alpha-1)\psi} = (1-\varphi)\,\alpha\,(\kappa^*)^{\psi-1} \left(\frac{\widehat{k}_t}{\widehat{x}_t}\right)^{(\alpha-1)\psi} = r^* + \delta.$$

This shows that the ratio between $\widehat{k}_t$ and $\widehat{x}_t$ must be constant over time, or equivalently,

$$\frac{\widehat{x}_{t+1}}{\widehat{x}_t} = \frac{\widehat{k}_{t+1}}{\widehat{k}_t} = \frac{\gamma^*}{1+a} = \frac{(1+q)(1-\tau^*)}{(1+a)(1+n)}.$$

By the same token, we can also rewrite (44) and (45) as

$$p_t = (1-\varphi)(1-\alpha)(\kappa^*)^{\psi-1} \left(\frac{\widehat{k}_t}{\widehat{x}_t}\right)^{(\alpha-1)\psi+1} Q_t, \tag{46}$$

$$w_t = \varphi\,(\kappa^*)^{\psi-1}\,\widehat{k}_t^{1-\psi} A_t. \tag{47}$$

Combining (46) and (4) gives

$$\frac{p_{t+1}}{p_t} = 1 + r^* = \frac{Q_{t+1}}{Q_t} = 1 + q.$$

The last step is to substitute (46) and (47) into the capital market clearing condition. This will give

$$K_{t+1} = A_t N_t \left(\kappa^*\right)^{\psi-1} \left[\frac{\varphi}{2+\theta}\widehat{k}_t^{1-\psi} - \left(\frac{1-\tau^*}{\tau^*}\right)(1-\varphi)(1-\alpha)\left(\frac{\widehat{k}_t}{\widehat{x}_t}\right)^{(\alpha-1)\psi+1}\widehat{x}_t\right]$$

$$\Rightarrow (1+a)(1+n)\,\widehat{k}_{t+1} = \left(\kappa^*\right)^{\psi-1} \left[\frac{\varphi}{2+\theta}\widehat{k}_t^{1-\psi} - \left(\frac{1-\tau^*}{\tau^*}\right)(1-\varphi)(1-\alpha)\left(\frac{\widehat{k}_t}{\widehat{x}_t}\right)^{(\alpha-1)\psi}\widehat{k}_t\right]$$

$$\Rightarrow (1+a)(1+n)\,\frac{\widehat{k}_{t+1}}{\widehat{k}_t} = \left(\kappa^*\right)^{\psi-1} \left[\frac{\varphi}{2+\theta}\widehat{k}_t^{-\psi} - \left(\frac{1-\tau^*}{\tau^*}\right)(1-\varphi)(1-\alpha)\left(\frac{\widehat{k}_t}{\widehat{x}_t}\right)^{(\alpha-1)\psi}\right].$$

Since both $\widehat{k}_{t+1}/\widehat{k}_t$ and $\widehat{k}_t/\widehat{x}_t$ are constant over time, it follows that the *level* of $\widehat{k}_t$ must be constant over time in any balanced growth equilibrium. Hence, we have

$$\frac{\gamma^*}{1+a} = \frac{(1+q)(1-\tau^*)}{(1+a)(1+n)} = 1.$$

Specification 2 Consider the production function in (33). The first-order conditions for the firm's problem are now given by

$$(1-\varphi)\,\alpha Y_t^{1-\psi} K_t^{\alpha\psi-1}\left(A_t N_t\right)^{(1-\alpha)\psi} = r_t + \delta,$$

$$\varphi Y_t^{1-\psi}\left(Q_t X_t\right)^{\psi-1} Q_t = p_t,$$

$$(1-\varphi)(1-\alpha)\,Y_t^{1-\psi} K_t^{\alpha\psi}\left(A_t N_t\right)^{\psi(1-\alpha)-1} A_t = w_t.$$

Using the two conditions: $Y_t = \frac{1}{\kappa^*}K_t$ and $r_t = r^*$, we can rewrite these as

$$(1-\varphi)\left(\kappa^*\right)^{\psi-1}\alpha\widehat{k}_t^{(\alpha-1)\psi} = r^* + \delta,$$

$$\varphi\left(\kappa^*\right)^{\psi-1}\left(\frac{\widehat{k}_t}{\widehat{x}_t}\right)^{1-\psi} Q_t = p_t, \tag{48}$$

$$(1-\varphi)(1-\alpha)\left(\kappa^*\right)^{\psi-1}\widehat{k}_t^{(\alpha-1)\psi+1} A_t = w_t.$$

The first one of these equations immediately implies that $\widehat{k}_t$ is constant over time, so that $\gamma^* = 1+a$. Substituting the last two equations into the capital market clearing condition gives

$$K_{t+1} = A_t N_t \left(\kappa^*\right)^{\psi-1} \left[\frac{\left(1-\varphi\right)\left(1-\alpha\right)}{2+\theta}\widehat{k}_t^{(\alpha-1)\psi+1} - \left(\frac{1-\tau^*}{\tau^*}\right)\varphi\left(\frac{\widehat{k}_t}{\widehat{x}_t}\right)^{1-\psi}\widehat{x}_t\right]$$

$$\Rightarrow \left(1+a\right)\left(1+n\right)\widehat{k}_{t+1} = \left(\kappa^*\right)^{\psi-1}\left[\frac{\left(1-\varphi\right)\left(1-\alpha\right)}{2+\theta}\widehat{k}_t^{(\alpha-1)\psi+1} - \left(\frac{1-\tau^*}{\tau^*}\right)\varphi\widehat{k}_t^{1-\psi}\widehat{x}_t{}^{\psi}\right].$$

Since $\widehat{k}_t$ is constant over time, the above equation implies that $\widehat{x}_t$ must be constant over time as well. Finally, (48) implies that p_t is growing at the same rate as Q_t in any balanced growth equilibrium so that $r^* = q$.

Specification 3 Next, we consider the production function in (34). The equilibrium factor prices are now characterised by

$$\left(1-\beta\right)\varphi\left[\varphi\widehat{k}_t^{\psi} + \left(1-\varphi\right)\widehat{x}_t^{\psi}\right]^{\frac{1-\beta}{\psi}-1}\widehat{k}_t^{\psi-1} = r_t + \delta, \tag{49}$$

$$\left(1-\beta\right)\left(1-\varphi\right)\left[\varphi\widehat{k}_t^{\psi} + \left(1-\varphi\right)\widehat{x}_t^{\psi}\right]^{\frac{1-\beta}{\psi}-1}\widehat{x}_t^{\psi-1}Q_t = p_t, \tag{50}$$

$$\left[\varphi\widehat{k}_t^{\psi} + \left(1-\varphi\right)\widehat{x}_t^{\psi}\right]^{\frac{1-\beta}{\psi}}\beta A_t = w_t. \tag{51}$$

The condition $Y_t = \frac{1}{\kappa^*}K_t$ can now be expressed as

$$\left[\varphi\widehat{k}_t^{\psi} + \left(1-\varphi\right)\widehat{x}_t^{\psi}\right]^{\frac{1-\beta}{\psi}} = \frac{1}{\kappa^*}\widehat{k}_t$$

Using this, we can rewrite (49)-(51) as

$$\left(1-\beta\right)\varphi\left(\kappa^*\right)^{\frac{\psi}{1-\beta}-1}\widehat{k}_t^{-\frac{\beta\psi}{1-\beta}} = r_t + \delta,$$

$$\left(1-\beta\right)\left(1-\varphi\right)\left(\kappa^*\right)^{\frac{\psi}{1-\beta}-1}\widehat{k}_t^{1-\frac{\psi}{1-\beta}}\widehat{x}_t^{\psi-1}Q_t = p_t,$$

$$\frac{1}{\kappa^*}\beta A_t\widehat{k}_t = w_t.$$

The first of these three equations, together with $r_t = r^*$, implies that $\widehat{k}_t$ must be constant over time. Hence, $\gamma^* = 1+a$. Substituting the last two equations into the capital market clearing condition gives

$$K_{t+1} = A_t N_t \left[\frac{\beta \widehat{k}_t}{(2+\theta)\,\kappa^*} - \left(\frac{1-\tau^*}{\tau^*} \right) (1-\beta)\,(1-\varphi)\,(\kappa^*)^{\frac{\psi}{1-\beta}-1}\, \widehat{k}_t^{1-\frac{\psi}{1-\beta}} \widehat{x}_t^{\psi} \right]$$

$$\Rightarrow (1+a)\,(1+n)\,\widehat{k}_{t+1} = \frac{\beta \widehat{k}_t}{(2+\theta)\,\kappa^*} - \left(\frac{1-\tau^*}{\tau^*} \right) (1-\beta)\,(1-\varphi)\,(\kappa^*)^{\frac{\psi}{1-\beta}-1}\, \widehat{k}_t^{1-\frac{\psi}{1-\beta}} \widehat{x}_t^{\psi}.$$

Since $\widehat{k}_t$ is constant over time, the above equation implies that $\widehat{x}_t$ must be constant over time as well. The remaining results follow by the same line argument as in Specification 2.

Specification 4 Finally, we consider the production function in (35). Equations (10) and (11) can be rewritten as

$$(1-v)\,\varphi \widehat{x}_t^{v} \left(\varphi \widehat{k}_t^{\psi} + 1 - \varphi \right)^{\frac{1-v-\psi}{\psi}} \widehat{k}_t^{\psi-1} = r_t + \delta \tag{52}$$

$$v \frac{Y_t}{X_t} = v \widehat{x}_t^{v-1} \left(\varphi \widehat{k}_t^{\psi} + 1 - \varphi \right)^{\frac{1-v}{\psi}} Q_t = p_t. \tag{53}$$

The condition $Y_t = \frac{1}{\kappa^*} K_t$ implies

$$\widehat{x}_t^{v} \left(\varphi \widehat{k}_t^{\psi} + 1 - \varphi \right)^{\frac{1-v}{\psi}} = \frac{1}{\kappa^*} \widehat{k}_t. \tag{54}$$

Combining (52), (54) and $r_t = r^*$ gives

$$\frac{1}{\kappa^*} \frac{(1-v)\,\varphi \widehat{k}_t^{\psi}}{\varphi \widehat{k}_t^{\psi} + 1 - \varphi} = r^* + \delta$$

$$\Rightarrow (1-v)\,\varphi \widehat{k}_t^{\psi} = (r^* + \delta)\,\kappa^* \left(\varphi \widehat{k}_t^{\psi} + 1 - \varphi \right).$$

This implies that $\widehat{k}_t$ must be constant over time. Hence, $\gamma^* = 1 + a$. Equation (54) then implies that $\widehat{x}_t$ is also constant over time. Hence, $1 - \tau^* = (1+a)\,(1+n)\,/\,(1+q)$. Finally, equation (53) implies that p_t and Q_t must be growing at the same rate. Hence, $r^* = q$.

This completes the proof of Theorem 3.

Appendix C: Infinitely-Lived Consumers

In this appendix, we will show that the main arguments in the proof of Theorem 1 and Theorem 2 can also be applied to an environment with infinitely-lived consumers. As a result, an endogenous growth solution similar to the one in Agnani, Gutiérrez and Iza (2005) can be obtained when the elasticity of substitution of $G(\cdot)$ is identical to one. But if this elasticity is bounded away from one, then the common growth factor γ^* and interest rate r^* are solely determined by the growth rates of the exogenous technological factors (i.e., A_t and Q_t). This shows that the main results of Section 3 are not specific to the OLG framework.

Consider an economy that is populated by $H > 0$ identical households. Each household contains a growing number of identical, infinitely-lived consumers. The size of each household at time t is given by $N_t = (1+n)^t$, with $n > 0$. Since all households are identical, we can focus on the choices made by a representative household and normalise H (which is just a scaling factor) to one. The representative household solves the following problem:

$$\max_{\{c_t, K_{t+1}, M_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t N_t \frac{c_t^{1-\sigma}}{1-\sigma}$$

subject to the sequential budget constraint

$$N_t c_t + K_{t+1} + p_t M_{t+1} = w_t N_t + (1+r_t) K_t + p_t M_t,$$

where $\beta \in (0,1)$ is the subjective discount factor; $\sigma > 0$ is the reciprocal of the elasticity of intertemporal substitution (EIS); c_t denotes individual consumption at time t; K_t and M_t are, respectively, the household's holding of physical capital and non-renewable resources; p_t, w_t and r_t are as defined in Section 2.1. The first-order conditions of this problem imply the Euler equation for consumption,

$$\frac{c_{t+1}}{c_t} = [\beta(1+r_{t+1})]^{\frac{1}{\sigma}}, \tag{55}$$

and the Hotelling rule,

$$\frac{p_{t+1}}{p_t} = 1 + r_{t+1}.$$

The rest of the economy is the same as in Sections 2.2 and 2.3. In particular, the first-order conditions for the firm's problem, (10)-(12), and the dynamic equation for natural resources, (13), remain unchanged. In any competitive equilibrium, goods market clear in every period so that

$$N_t c_t + K_{t+1} - (1-\delta)K_t = F\left(K_t, G\left(Q_t X_t, A_t N_t\right)\right), \qquad \text{for all } t \geq 0. \tag{56}$$

This replaces the capital market clearing condition that we use in Step 3 of the proof of Theorem 1 and Theorem 2 in the OLG economy.

When characterising a balanced growth equilibrium, we maintain the four conditions (v)-(viii) listed in Section 3. First consider the case when $G(\cdot)$ takes the Cobb-Douglas form, or equivalently, $\sigma_G(\cdot)$ is identical to one. Dividing both sides of (56) gives

$$\frac{N_t c_t}{K_t} + \frac{K_{t+1}}{K_t} - (1-\delta) = \frac{F\left(K_t, G\left(Q_t X_t, A_t N_t\right)\right)}{K_t}.$$

Hence, in any balanced growth equilibrium, aggregate consumption $N_t c_t$ must be growing at the same rate as K_t and Y_t. This, together with the Euler equation in (55) implies

$$\gamma^* = \left[\beta\left(1+r^*\right)\right]^{\frac{1}{\sigma}},$$

where γ^* is again the growth factor of per-capita output in a balanced growth equilibrium. Next, note that the arguments in Step 1 and Step 2 of the proof of Theorem 1 are built upon the properties of the production function and the characterising properties of balanced growth equilibrium. In particular, these arguments do not rely on the consumer side of the economy. Hence, they remain valid in this environment. Consequently, we have

$$\gamma^* = (1+b)\left(\frac{1-\tau^*}{1+n}\right)^{1-\phi},$$

$$\left(1+r^*\right)\left(1-\tau^*\right) = \gamma^*\left(1+n\right),$$

where $1+b \equiv (1+a)^{\phi}(1+q)^{1-\phi}$. Using these three equations, we can derive

$$1+r^* = \beta^{-\frac{\phi}{\varpi}}\left(1+b\right)^{\frac{\sigma}{\varpi}},$$

$$1-\tau^* = \beta^{\frac{1}{\varpi}}\left(1+b\right)^{\frac{1-\sigma}{\varpi}}\left(1+n\right),$$

$$\gamma^* = \beta^{\frac{1-\phi}{\varpi}}\left(1+b\right)^{\frac{\sigma}{\varpi}},$$

where $\varpi \equiv 1-(1-\sigma)(1-\phi)$. Thus, a unique balanced growth equilibrium exists if

$$\beta^{\frac{1}{\varpi}}(1+b)^{\frac{1-\sigma}{\varpi}}(1+n) \in (0,1),$$

which ensures that $\tau^* \in (0,1)$. Notice that both γ^* and τ^* are endogenously determined by a host of factors as in the AGI solution.

Suppose now $\sigma_G(\cdot)$ is never equal to one. Since the arguments in Step 1 and Step 2 of the proof of Theorem 2 remain valid in this environment, we have $\gamma^* = 1+a$, $r^* = q$, $\widehat{k}_t = \widehat{k}^*$ and $\widehat{x}_t = \widehat{x}^*$. These in turn imply that

$$1-\tau^* = \frac{(1+a)(1+n)}{1+q}.$$

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