UNIVERSITY OF LEICESTER

School of Business
Economics Division

Skill Premium and Technological Change in the Very Long Run: 1300-1914

Rui Luo, University of Leicester

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Rui Luo
University of Leicester
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Abstract

This paper sets out to explain the historical development of the skill premium in western Europe over a period ranging from the pre-modern era to the modern era (circa 1300 to 1914). We develop a model of the skill premium and technological change over the very long run which endogenously accounts for the transition across different growth regimes in this period. The model integrates two key elements in long-run growth, the human capital investment and the capital-human capital ratio, into the analysis and successfully explains the declining skill premium from 1300 to 1600 and the stable skill premium from 1600 to 1914. The explanation elucidates a number of well-known historical facts that have not been previously examined in the study of the skill premium.

Key Words: skill premium, technological change, human capital investment, capital-human capital ratio, growth regimes

JEL Classification: J31, O41, O11

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1. Introduction

This paper studies the evolution of the skill premium in Western Europe from 1300 to 1914. The skill premium is the ratio of wage of skilled labour to that of unskilled labour. Its evolution over time is closely related to technological change and economic development. We know that the economic development in western Europe in this period is representative of technological change and development in the very long run. Analysing the evolution of the skill premium in western Europe in the period from 1300 to 1914 will uncover how technological progress and economic development contribute to the formation of the skill premium in the very long run.

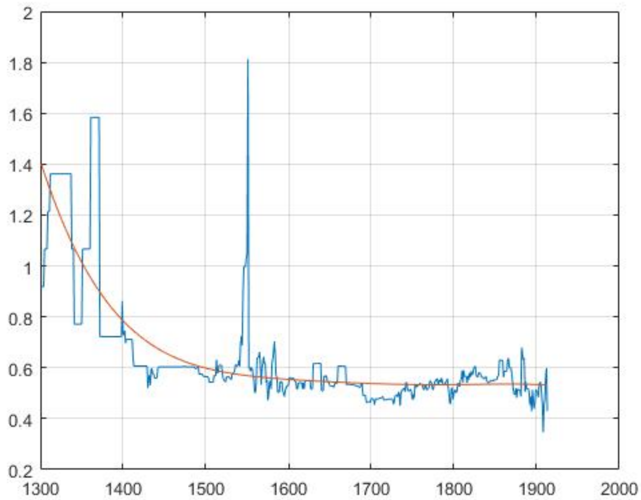

FIGURE 1. Skill Premium in Western Europe from 1300 to 1914 [data source: estimation of wages made by Allen (2001)].

Figure 1 depicts the evolution of the skill premium in western Europe.[2] It can be seen that the skill premium declines sharply from approximately 120% to approximately 58% from 1300 to 1600, then remains stable at a level of approximately 58% from 1600 to 1914. This shows that the economic development and technological progress lead the skill premium to decrease and then converge to a stable level. By contrast, on contemporary days, the skill premium increases along with economic development and technological progress. For instance, the skill premium in U.S. significantly goes up from 1980 to the mid-1990s, a period that sees accelerating technological change and growing relative supply of skilled labour(Figure 1 in Acemoglu (2002)). It would then be interesting to examine why economic development and technological change in the past results in a stable pattern of the skill premium, while those at present give rise to an increasing trend of the skill premium.

[2]Skill premium here is calculated by the ratio of wage of skilled craftsmen and masons to that of unskilled labourers minus 1. The area of "Western Europe" consists of five cities: London, Oxford, Amsterdam, Antwerp and Paris.

The phenomenon that the skill premium exhibits an increasing trend in contemporary days has been thoroughly studied by the "skill-biased technological change"(SBTC) literature. Acemoglu (2002) develops a model showing that increasing relative supply of skilled labour induces SBTC, which leads to higher relative demand for skilled labour. As a result, the skill premium increases. The SBTC model well explains the pattern of the skill premium in the post-modern period in which rapid technological change favours the skilled labour more than the unskilled one. In the very long run (circa 1300 to 1914), however, the economic growth and the technological change are different from those in post-1970s U.S. Plus, the pattern of economic growth varies in different periods of history. Thus we need a unified growth model similar to the one in the long-run growth literature so as to incorporate the growth and technological change in different epochs into the analysis of the skill premium in the very long run. On the basis of the unified growth models developed by Galor and Moav (2004) and Galor and Weil (2000), we develop a model that characterizes the growth and the technological change in the very long run. And we show that the technological change in this long historical period first lowers down the skill premium at the beginning, then acts as a "stabilizer" of the skill premium afterwards. Only when SBTC or the ability-biased technological change (Galor and Tsiddon (1997), Galor and Moav (2000)) takes place will the skill premium increase. This reaffirms the key role that SBTC plays in widening wage inequality. To the best of our knowledge, this paper is the first to analytically study the evolution of the skill premium in the very long run in a unified framework of growth.[3]

The key finding in this paper is that the evolution of the skill premium is governed by two factors. One is the growth of the human capital investment and the other is the growth of the relative abundance of physical capital relative to human capital measured by the capital-human capital ratio. Galor and Moav (2004) show that the growth of the capital-human capital ratio not only drives the economic growth and development from the primitive stage to the advanced stage, but also determines whether higher initial income inequality stimulates or inhibits the economic development. Nevertheless, the initial income inequality in their analysis is exogenously given. Our finding shows that in addition to influencing the economic development, the growth of the capital-human capital ratio affects the degree of wage inequality as well.[4] This is new to the existing literature on long-run growth.

The growth of the human capital investment and that of capital-human capital ratio have competing effect on the skill premium. Increasing human capital investment pushes the skill premium upward in two channels: on one hand, growing human capital investment incurs the productivity of human capital to grow faster than that of unskilled worker.

[3]There are studies on the evolution of the skill premium in this period. For example, van Zanden (2009) tries to explain why the skill premium in western Europe evolves in such pattern besides numerical analysis. But the explanations are intuitive and qualitative.

[4]van Zanden (2009) has a similar finding. But the variable he concerns about is the capital/land-labour ratio, not the capital-human capital ratio.

Similar to Acemoglu (2002), this makes technological change more biased towards skilled labour. increasing its relative marginal product.[5] On the other hand, more human capital investment raises both the return and the cost of becoming skilled labour. The return to becoming a skilled worker increases in a diminishing manner due to diminishing return to scale. The cost of becoming skilled labour, however, increases linearly as a result of growing human capital investment, which overwhelms the increase in return. The net return to becoming skilled labour decreases, discouraging individuals from becoming skilled worker. Lower relative supply of skilled labour and the increasing relative demand for skilled labour jointly raise the skill premium. Growing capital-human capital ratio, however, has a negative effect on the skill premium. Higher capital-human capital ratio, while increasing the return to becoming skilled labour (i.e. the wage of skilled labour goes up), reduces the cost of becoming skilled labour by lowering down the interest rate. The net return to becoming skilled labour increases, incurring more people to invest in human capital and work as skilled workers. In this way, growing capital-human capital ratio reduces the skill premium.

Which one of the two competing effects dominates the other depends on the level of the capital-human capital ratio. This paper shows that when capital-human capital ratio is low (i.e. it is below certain threshold), the negative effect of capital-human capital ratio dominates. When the capital-human capital ratio becomes higher (i.e. it goes beyond the threshold), the negative effect of the growing capital-human capital ratio and the positive effect of the growing human capital investment cancel out. As the capital-human capital ratio becomes sufficiently high, the negative effect of the growing capital-human capital ratio becomes dominant again.

The economic development and transition from the pre-modernity to modernity is driven by growing capital-human capital ratio, which is augmented by technological progress. As discussed before, different levels of capital-human capital ratio result in three different scenarios, two of which see the negative effect of capital-human capital ratio dominating the positive effect of human capital investment and one of which sees the two effects cancelled out. This variation in which effect dominates the other partitions the process of development into three different regimes of growth. They can be referred to as: the late medieval regime (circa 1300 to 1600), featured with low capital-human capital ratio and equivalent to the primitive stage of development, the early modern regime (circa 1600 to 1800), featured with higher capital-human capital ratio and equivalent to an intermediate stage of development, and the modern growth regime (circa 1800 to 1914), featured with sufficiently high capital-human capital ratio and equivalent to the advanced stage of development. These regimes constitute a unified growth framework similar to the unified growth models in Galor and Weil (2000) and Galor and Moav (2004).

[5] Technological change spurred by growing human capital investment is biased towards the skilled labour because an increase in the ratio of the productivity of the skilled labour to that of the unskilled labour raises the skill premium. Acemoglu (2002) draws similar conclusion.

In this unified growth framework, we find that in the late medieval regime (1300-1600), the capital-human capital ratio is so low that the negative effect of capital-human capital ratio dominates. As the capital-human capital ratio slowly increases due to slow technological change, the skill premium goes down. This corresponds to the "declining part" of the skill premium from 1300 to 1600 in Figure I. In the early modern regime (1600-1800), the capital-human capital ratio becomes higher so that its negative effect counteracts the positive effect of human capital investment. The skill premium stays the same while capital-human capital ratio continues to grow as a result of slow technological change. Eventually, as the capital-human capital ratio becomes sufficiently high, the economy takes off into the modern growth regime (after 1800), which sees higher rate of technological progress. This makes it more profitable to become skilled worker, which raises human capital investment. The positive effect of human capital investment thus gains an initial domination, causing an upward jump to the skill premium. On the other hand, sufficiently high capital-human capital ratio causes the negative effect of capital-human capital ratio to become dominant. Then as capital-human capital ratio continues to grow, the skill premium goes down and converges back to the same level as in the previous regime in the long run. This trajectory of the evolution of the skill premium from 1600 to 1914 proposed by our model corresponds to the stable part of the skill premium in the same period in Figure I.

Because the driving force of the growth of the capital-human capital ratio is technological change, our findings then indicate that it is the technological change in the past that contributes to the "first declining then stable" pattern of the evolution of the skill premium. That is, technological progress in the past balances the relative demand for skilled labour with its relative supply. Only when contemporary SBTC or the ability-biased technological change (Galor and Tsiddon (1997), Galor and Moav (2000)) will possibly raise the skill premium by creating excessive relative demand for skilled labour. This reaffirms the key role that contemporary SBTC plays in widening wage inequality.

In addition to what is mentioned before, this paper makes other contributions to the long-run growth literature. Studies on the interaction between inequality and growth in the very long run have been carried out by Galor and Zeira (1993), Galor and Moav (2004) and Galor et. al (2009). But they examine how variation in inequality affects the outcome of development. By comparison, we examine how growth and technological change in the very long run shape the skill premium. This adds new insights to existing research by examining a reversed direction of the causal relation between inequality and growth.

The unified framework of growth in this paper, while inheriting features of their counterparts Galor and Weil (2000) and Galor and Moav (2004), vary to some extent to capture a more realistic picture of growth and development. The late medieval regime (1300-1600) in our framework shares the feature of inactive human capital investment and slow technological progress with the "Malthusian epoch" in Galor and Weil (2000). Yet the human capital investment in our late medieval regime is positive and fixed at a low but positive level, while that in the "Malthusian epoch" is zero. This modification makes it possible to calculate the skill premium even in primitive stage of development[6]. While Galor and Weil (2000) categorize the period from 1600 to 1800 as the "Malthusian epoch" featured with zero human capital investment, we characterize this period, the early modern regime in our framework, with mild growth of human capital investment. Deviate from Galor and Weil (2000) as it appears to, this modification captures the recent findings which indicate that the early modern period sees mild growth instead of Malthusian stagnation. For instance, a recent empirical finding made by Broadberry et al (2015), who state that "successful" economies in western Europe already grow beyond the level of "bare-bone subsistence" level. Another recent study on long-run growth made by Foreman-Peck and Zhou show that human capital in England already started growing in this period[7]. Nevertheless, technological progress in the early modern regime follows a slow pattern similar to the "Malthusian epoch" in Galor and Weil (2000), indicating this regime still belongs to pre-modern period. As for the modern growth regime in our framework (circa after 1800), it shares the similar feature of fast and sustainable growing technology (i.e. technological progress is augmented by human capital investment) with its counterpart in Galor and Weil (2000). Generally speaking, the unified framework of growth our analysis on the skill premium is based upon is fundamentally similar to its counterpart in canonical long-run growth literature.

Deeply rooted in existing long-run growth literature, this paper proposes a more powerful explanation on the skill premium than existing studies. This can be seen in explaining "declining part" of the premium. Previously, van Zanden (2009) attributes this to the demographic decline left by the Black Death. He proposes that smaller population results in lower return to capital investment[8], which increases individuals' incentive to invest in human capital. This raises the relative supply of skilled labour and human capital, resulting in a drop in the skill premium.[9] Had this been true, households would have "actively" invested in human capital. However, the canonical unified growth theory developed by Galor and Weil (2000) suggests inactive human capital investment in this period. Galor and Ashraf (2013) demonstrate that the demographic decline results in "a larger but not significantly richer" population. Population recovers while human capital investment

[6]The skill premium in Figure 1 is calculated by deducting the ratio of the wage of skilled labour to that of unskilled labour by 1. This indicates that the wage of skilled labour is higher than that of unskilled labour. On the other hand, zero human capital investment implies zero cost of becoming skilled labour. This leads to identical wages for skilled and unskilled labour and the skill premium will be zero. A contradiction with Figure 1.

[7]This comes from a paper entitled "Bring Unified Growth Model to the Data", which was presented in the Royal Economic Society Annual Conference in 2016. Further details of this paper can be found on this website: `http://www.res.org.uk/details/mediabrief/9077771/LATER-MARRIAGES-PLAYED-A-KEY-ROLE-IN-EUROPES-HISTORIC-GROWTH-TAKE-OFF.html`

[8]According to van Zanden (2009), this is because capital-labour ratio increases after the demographic decline.

[9]Even though the demographic decline may incur changes that stir the rise of "modern Europe" (See Pamuk (2007)), which may increase the demand for skills as well, van Zanden (2009) argues that the increase in demand is weaker than in supply.

stays low. This seems at odds with the "active" human capital investment hypothesis in van Zanden (2009). This paper, on the contrary, highlights the role that inactive human capital investment plays in causing declining part of the skill premium in Figure 1: The inactive human capital investment makes the negative effect of growing capital-human capital ratio dominate. We also show that the declining skill premium is a natural result of increasing capital-skilled labour ratio and the exogenous demographic decline is not necessary for such decline to occur. In this way, this paper effectively explains the declining skill premium without rejecting the fundamental features of the epoch in which the decline happens.

The rest of the paper is organized as follows: Section 2 describes the model and Section 3 shows the formation of different epochs of growth; Section 4 studies the formation the skill premium in different epochs and Section 5 concludes.

2. The Outline of the Model and the Process of Development

2.1. **Production.** Consider an economy which contains two sectors. The first one is the skilled labour-intensive sector, with output denoted as Y^{S}. The second one is the unskilled labour-intensive sector, with output denoted as Y^{U}. Aggregate output at time t, Y_t, is the summary of the outputs from both sectors.

$$Y_t = Y_t^{\mathrm{S}} + Y_t^{\mathrm{U}} \tag{1}$$

In period t, skilled labour-intensive sector hires physical capital K_t and effective human capital $\tilde{H}_t$ for production. $\tilde{H}_t$ satisfies $\tilde{H}_t = S_t A_t h_t$. S_t denotes the supply of skilled labour, which can be seen as the "skilled craftsman" in the context of economic history. h_t denotes the amount of human capital supplied by each skilled worker, which depreciates at the end of the period. A_t denotes the knowledge of each skilled worker, which carries on forever once it is generated. The production has constant returns to scale with respect to K_t and $\tilde{H}_t$:

$$Y_t^{\mathrm{S}} = K_t^{\alpha}(S_t A_t h_t)^{1-\alpha} = K_t^{\alpha}(\tilde{H}_t)^{1-\alpha} \tag{2}$$

According to (2), A_t can be seen as the productivity of skilled labour. And the skilled-intensive sector hires skilled labour, human capital with capital for production, it is thus equivalent to a sector of "industry". In this way, the growth of A_t can be seen as the growth of the productivity of the industry. The development of this sector drives the economy towards the era of industrialization.

Unskilled-intensive sector hires unskilled labour U_t and $\bar{X}$ for production. $\bar{X}$ is a fixed input factor and is usually referred to as land. The unskilled-intensive sector is similar to the "agricultural sector" and its output is formulated as:

$$Y_t^{\mathrm{U}} = \bar{X}^{\alpha}(A_t^{\mathrm{U}} U_t)^{1-\alpha} \tag{3}$$

A_t^U denotes the productivity of unskilled labour at time t. Its growth reflects the growth of agricultural productivity. The unskilled labour U_t is equivalent to the "labourers" in the context of economic history.

Denote human capital as $H_t = S_t h_t$. We can then define capital-effective human capital ratio $\tilde{k}_t$

Definition 1. *Capital-effective human capital ratio $\tilde{k}_t$ is formulated as:*

$$\tilde{k}_t = \frac{K_t}{\tilde{H}_t} = \frac{K_t}{S_t A_t h_t} \tag{4}$$

And we can define capital-human capital ratio k_t as:

$$k_t = \frac{K_t}{H_t} = \frac{K_t}{S_t h_t} \tag{5}$$

(4) and (5) imply the following relation between $\tilde{k}_t$ and k_t

$$\tilde{k}_t = \frac{k_t}{A_t} \tag{6}$$

With capital-effective human capital ratio $\tilde{k}_t$, we can derive the inverse demand for effective human capital (wage per effective human capital) as:

$$\tilde{w}_t = (1-\alpha)\tilde{k}_t^{\alpha} \tag{7}$$

At equilibrium, the wage of skilled labour, unskilled labour and interest rate equal to the marginal product of skilled labour, unskilled labour and capital, respectively. We can then derive the interest rate r_t in terms of $\tilde{k}_t$ and k_t as:

$$r_t = \alpha K_t^{\alpha-1}(S_t A_t h_t)^{1-\alpha} = \alpha k_t^{\alpha-1} A_t^{1-\alpha} = \alpha \tilde{k}_t^{\alpha-1} \tag{8}$$

The equilibrium wage of skilled labour, w_t^{S}, can be written in terms of wage per effective human capital, $\tilde{w}_t$, as:

$$w_t^{\text{S}} = (1-\alpha)K_t^{\alpha} S_t^{-\alpha}(A_t h_t)^{1-\alpha} = (1-\alpha)\tilde{k}_t^{\alpha} A_t h_t = \tilde{w}_t A_t h_t \tag{9}$$

Also we can write the equilibrium wage of skilled labour in terms of capital-human capital ratio, k_t, as:

$$w_t^{\text{S}} = (1-\alpha)k_t^{\alpha} A_t^{1-\alpha} h_t \tag{10}$$

Combining (9) with (10) and we have:

$$w_{t+1}^{\text{S}} = \tilde{w}_t A_t h_t = (1-\alpha)k_t^{\alpha} A_t^{1-\alpha} h_t \tag{11}$$

Lastly the equilibrium wage of unskilled labour w_{t+1}^{U} can be written as:

$$w_t^{\text{U}} = (1-\alpha)\bar{X}^{\alpha}(A_t^{\text{U}})^{1-\alpha} U_t^{-\alpha} \tag{12}$$

2.2. **Individuals.** An individual i born at the beginning of period t lives for two periods, period t and period $t+1$. The first one is the period of childhood and the second one is the period of adulthood. Each individual has one parent and gives birth to one child.

In the period of childhood, the young individual, while inheriting an amount of bequest from his or her parent, decides whether to become skilled labour or not after growing up. If yes, the individual devotes part of the bequest to human capital investment and acquires knowledge and human capital. If not, the individual saves all the bequest and will work as an unskilled worker after growing up.

In the period of adulthood, the grown-up individual either works as a skilled worker or an unskilled worker depending on whether human capital investment was made in the period of childhood. The individual earns the wage plus the return to the net asset saved in the previous period. The aggregate income in this period is often referred to as the "second-period wealth" of the individual. The individual allocates part of the wealth to consumption and the rest to bequest transferred to the next generation.

2.2.1. *Individual's Second Period Wealth and Preference.* Denote the second period wealth of individual i as I^{i}_{t+1}. If the individual works as a skilled worker, he or she supplies h^{i}_{t+1} units of efficient labour and A^{i}_{t+1} units of knowledge. For each unit of the efficient labour and knowledge supplied, the individual earns the market wage $\tilde{w}_{t+1}$ formulated in (7). And he receives the return to the savings of net asset in the previous period, $x^{\mathrm{i,S}}_{t+1}$. So the second period wealth of the individual satisfies $I^{\mathrm{i}}_{t+1} = \tilde{w}_{t+1}A^{\mathrm{i}}_{t+1}h^{\mathrm{i}}_{t+1} + x^{\mathrm{S}}_{t+1}$.

If the individual works as an unskilled worker, he or she receives the wage of unskilled labour w^{U}_{t+1} formulated in (12) and the return to the savings of the net asset in the previous period x^{U}_{t+1}. And the second period wealth satisfies: $I^{\mathrm{i}}_{t+1} = w^{\mathrm{U}}_{t+1} + x^{\mathrm{U}}_{t+1}$.

As mentioned before, the funding for the human capital investment necessary to become skilled labour comes from the bequest. So the net asset a skilled worker has in the period of childhood is the parental bequest deducted by the amount of human capital investment. And for an unskilled worker, the net asset in the period of childhood equals to the parental bequest. Assume that the amount of bequest individual i inherits from parent in childhood is b^{i}_t, the amount of human capital investment is e^{i}_t and the rate of return to savings is R_{t+1}. Then the return to the net asset given the individual is a skilled worker, x^{S}_{t+1}, and that given the individual is an unskilled worker, x^{U}_{t+1}, satisfy:

$$\begin{aligned} x^{\mathrm{S}}_{t+1} &= (b^{\mathrm{i}}_t - e^{\mathrm{i}}_t)R_{t+1} \\ x^{\mathrm{U}}_{t+1} &= b^{\mathrm{i}}_t R_{t+1} \end{aligned} \tag{13}$$

In (13), R_{t+1} is the aggregate rate of return to the net asset saved in the previous period. We have $R_{t+1} = 1 + r_{t+1} - \delta$, where r_{t+1} is the interest rate and δ is the depreciation rate. Following Galor and Moav (2004) we assume full depreciation of capital so that $\delta = 1$. So we have $R_{t+1} = r_{t+1}$.

Using (13) and $R_{t+1} = r_{t+1}$, we can write the second period wealth of the individual as:

$$I^{\mathrm{i}}_{t+1} = \begin{cases} \tilde{w}_{t+1}A^{\mathrm{i}}_{t+1}h^{\mathrm{i}}_{t+1} + (b^{\mathrm{i}}_t - e^{\mathrm{i}}_t)r_{t+1} \equiv I^{\mathrm{i,S}}_{t+1} & \text{if} \quad e^{\mathrm{i}}_t > 0 \\ w^{\mathrm{U}}_{t+1} + b^{\mathrm{i}}_t r_{t+1} \equiv I^{\mathrm{i,U}}_{t+1} & \text{if} \quad e^{\mathrm{i}}_t = 0 \end{cases} \tag{14}$$

In (14), $I_{t+1}^{\mathrm{i,S}}$ and $I_{t+1}^{\mathrm{i,U}}$ are notations of the second period wealth given that the individual works as a skilled worker and an unskilled worker respectively.

Now we define the individual's preference. Similar to Galor and Moav (2004), the preference of individual i who was born at the beginning of period t is reflected by a trade off between consumption in period $t+1$, c_{t+1}^{i}, and bequest for the next generation born at the beginning of period $t+1$, b_{t+1}. We use a log linear function similar to Galor et al (2009) to formulate individual's lifetime utility u_t^{i}:

$$u_t^{\mathrm{i}} = (1-\beta)\log c_{t+1}^{\mathrm{i}} + \beta \log b_{t+1}^{\mathrm{i}} \tag{15}$$

$\beta \in (0,1)$ holds. The individual maximizes the objective function (15) subject to the following budget constraint:

$$c_{t+1}^{\mathrm{i}} + b_{t+1}^{\mathrm{i}} \leq I_{t+1}^{\mathrm{i}} \tag{16}$$

So the optimal consumption will be:

$$c^{\mathrm{i}} = (1-\beta) I_{t+1}^{\mathrm{i}} \tag{17}$$

And the optimal bequest to be transferred to the next generation will be:

$$b_{t+1}^{\mathrm{i}} = \beta I_{t+1}^{\mathrm{i}} \tag{18}$$

If we plug (17) and (18) into (15), we can derive the indirect second period utility function for individual i, V_{t+1}^{i}, as:

$$V_{t+1}^{\mathrm{i}} = \log[(1-\beta)^{1-\beta}\beta^{\beta}] + \log I_{t+1}^{\mathrm{i}} \tag{19}$$

Equation (19) shows that the second period utility of the grown-up individual i is increasing with respect to the second period wealth I_{t+1}^{i}. So for individual i, maximizing lifetime utility u_t^{i} is equivalent to maximizing the second period wealth I_{t+1}^{i}.

2.2.2. *Human Capital Investment Decision.* As discussed before, individual i spends e_t^{i} on human capital investment in his or her childhood in order to become skilled labour. In return to this, individual i supplies h_{t+1}^{i} units of efficient labour after growing up. And the "baseline" level of the efficient labour is unity (i.e. $h_{t+1}^{\mathrm{i}} = 1$). We formulate the supply of efficient labour as a function of human capital investment as follows:

$$h_{t+1}^{\mathrm{i}} = h(e_t^{\mathrm{i}}) = \begin{cases} 1 & \underline{e} \leq e_t^{\mathrm{i}} \leq 1 \quad \underline{e} \in (0,1) \\ (e_t^{\mathrm{i}})^{\gamma} & e_t^{\mathrm{i}} > 1 \quad \gamma \in (0,1) \end{cases} \tag{20}$$

According to (20), the efficient labour acquired is restricted to the level of unity for small human capital investment (i.e. $e_t^{\mathrm{i}} \leq 1$). When e_t^{i} is high enough (i.e. $e_t^{\mathrm{i}} > 1$), the efficient labour obtained is an increasing function of human capital investment e_t^{i} with decreasing marginal return to human capital investment. To the best of our knowledge, none of the relevant studies have adopted the formulation of the human capital production as in (20). Nevertheless, this formulation of human capital production in (20) shares one fundamental function with its counterpart in literature such as Galor and Moav (2004).

That is, the interior solution for the optimal amount of human capital investment does not always exist. When the amount is small, there exist a corner solution for the optimal human capital investment. This is crucial to partitioning the process of development into different regimes.

Human capital investment in t, e_t^{i}, not only generates efficient labour, h_{t+1}^{i}, but also augment the growth of the knowledge stock A_{t+1}^{i}. The growth rate of A_{t+1}^{i}, g_{t+1}^{i}, is assumed as follows:

Assumption 1. *The growth rate of the knowledge stock of a skilled worker, g_{t+1}^{i}, is an increasing function of human capital investment e_t^{i}. It is formulated as:*

$$g_{t+1}^{i} = g(e_t^{i}) = \begin{cases} \bar{g}\left[1 - \frac{1}{(e_t^{i})^{\gamma}}\right] - 1 & e_t > \left(\frac{\bar{g}}{\bar{g}-1}\right)^{\frac{1}{\gamma}} \\ 0 & \textit{otherwise} \end{cases} \tag{21}$$

The parameter $\bar{g}$ in equation (21) satisfies $\bar{g} > 1$.

According to assumption 1, human capital investment does not augment the growth of knowledge stock per skilled worker unless its growth rate, which is generated by human capital investment, is positive.

Now we can derive the optimal amount of human capital investment. We can write the individual's second period wealth given that he or she works as a skilled worker as:

$$I_{t+1}^{\mathrm{i,S}} = \tilde{w}_{t+1}A_{t+1}^{\mathrm{i}}h_{t+1}^{\mathrm{i}} + (b_t^{\mathrm{i}} - e_t^{\mathrm{i}})r_{t+1} = \tilde{w}_{t+1}A_t^{\mathrm{i}}(1 + g(e_t^{\mathrm{i}}))h(e_t^{\mathrm{i}}) + (b_t^{\mathrm{i}} - e_t^{\mathrm{i}})r_{t+1} \tag{22}$$

The individual's human capital investment is aimed at maximizing the lifetime utility u_t^{i}. As equation (19) shows, the higher the second period wealth is, the higher the lifetime utility will be. So optimal human capital investment $(e_t^{\mathrm{i}})^*$ should maximize the second period income formulated in (22). So $(e_t^{\mathrm{i}})^*$ satisfies:

$$(e_t^{\mathrm{i}})^* = \mathrm{argmax}[\tilde{w}_{t+1}A_t^{\mathrm{i}}(1 + g(e_t^{\mathrm{i}}))h(e_t^{\mathrm{i}}) + (b_t^{\mathrm{i}} - e_t^{\mathrm{i}})r_{t+1}] \tag{23}$$

The formulation of $h(e_t^{\mathrm{i}})$ and $g(e_t^{\mathrm{i}})$ indicate that there are three different intervals which human capital investment e_t^{i} falls into: 1) $e_t^{\mathrm{i}} \leq 1$; 2) $1 < e_t^{\mathrm{i}} \leq \bar{g}/(\bar{g}-1)$ and 3) $e_t^{\mathrm{i}} > \bar{g}/(\bar{g}-1)$. In the first two cases, $g(e_t^{\mathrm{i}}) = 0$ holds, which means that the stock of knowledge A_{t+1} is constant. We normalize it to 1. We now derive $(e_t^{\mathrm{i}})^*$ in each case.

In the first case, (20) implies $h(e_t^{\mathrm{i}}) = 1$. And because of $\bar{g}/(\bar{g}-1) > 1$, $e_t^{\mathrm{i}} \leq 1$ implies $e_t^{\mathrm{i}} < \bar{g}/(\bar{g}-1)$. According to (21), $g(e_t^{\mathrm{i}}) = 0$ holds in this case. And A_{t+1} is normalized to 1, as mentioned before. In this way, (23) can be written as:

$$(e_t^{\mathrm{i}})^* = \mathrm{argmax}[\tilde{w}_{t+1} + (b_t^{\mathrm{i}} - e_t^{\mathrm{i}})r_{t+1}] \tag{24}$$

$\tilde{w}_{t+1}$ and r_{t+1} denote the market levels of wage per effective human capital and interest rate. b_t^{i} is the amount of parental bequest. All of them are taken as given. Then (24) indicates that we should set e_t^{i} as close to zero as possible to maximize the second period wealth (hence the utility) of the individual. According to (20) the lowest level of e_t^{i} is $\underline{e}$. So optimal human capital investment satisfies:

$$(e_t^{\mathrm{i}})^* = \underline{e} \equiv e_t^{(1)} \tag{25}$$

In the second case, $1 < e_t^{\mathrm{i}} \leq \bar{g}/(\bar{g}-1)$ implies that $h(e_t^{\mathrm{i}}) = (e_t^{\mathrm{i}})^\gamma$ and $g(e_t^{\mathrm{i}}) = 0$ hold. And we have $A_{t+1} = 1$. Given these, we can rewrite (23) as:

$$(e_t^{\mathrm{i}})^* = \text{argmax}[\tilde{w}_{t+1}(e_t^{\mathrm{i}})^\gamma + (b_t^{\mathrm{i}} - e_t^{\mathrm{i}})r_{t+1}] \tag{26}$$

The first order condition of e_t^{i} derived from (26) is:

$$\tilde{w}_{t+1}\gamma(e_t^{\mathrm{i}})^{\gamma-1} = r_{t+1} \tag{27}$$

Equations (7) and (8) show that $\tilde{w}_{t+1}$ and r_{t+1} are functions of capital-effective human capital ratio $\tilde{k}_{t+1}$. Then from (27) we can derive optimal human capital investment $(e_t^{\mathrm{i}})^*$ as a function of $\tilde{k}_{t+1}$ as:

$$(e_t^{\mathrm{i}})^* = \left[\frac{\gamma(1-\alpha)}{\alpha}\tilde{k}_{t+1}\right]^{\frac{1}{1-\gamma}}$$

In this case we have $A_{t+1} = 1$. Then based on the relation between $\tilde{k}_{t+1}$ and capital-human capital ratio k_{t+1} in (6), we have $k_{t+1} = \tilde{k}_{t+1}A_{t+1} = \tilde{k}_{t+1}$. We can then write $(e_t^{\mathrm{i}})^*$ in terms of k_{t+1} as:

$$(e_t^{\mathrm{i}})^* = \left[\frac{\gamma(1-\alpha)}{\alpha}k_{t+1}\right]^{\frac{1}{1-\gamma}} \equiv e^{(2)}(k_{t+1}) \tag{28}$$

In the third case, $e_t^{\mathrm{i}} > \bar{g}/(\bar{g}-1) > 1$ implies that $g(e_t^{\mathrm{i}}) > 0$ and $h(e_t^{\mathrm{i}}) = (e_t^{\mathrm{i}})^\gamma$ hold. Then the optimal human capital investment is derived exactly from the formulation in (23).

Using (20) and (21) we can write the term $[1 + g(e_t^{\mathrm{i}})]h(e_t^{\mathrm{i}})$ in (23) as:

$$[1 + g(e_t^{\mathrm{i}})]h(e_t^{\mathrm{i}}) = \bar{g}\left[1 - \frac{1}{(e_t^{\mathrm{i}})^\gamma}\right](e_t^{\mathrm{i}})^\gamma = \bar{g}[(e_t^{\mathrm{i}})^\gamma - 1]$$

Then (23) can be written as:

$$(e_t^{\mathrm{i}})^* = \text{argmax}[\tilde{w}_{t+1}A_t^{\mathrm{i}}\bar{g}[(e_t^{\mathrm{i}})^\gamma - 1] + (b_t^{\mathrm{i}} - e_t^{\mathrm{i}})r_{t+1}] \tag{29}$$

The first order condition is:

$$\tilde{w}_{t+1}A_t^{\mathrm{i}}\bar{g}\gamma(e_t^{\mathrm{i}})^{\gamma-1} = r_{t+1} \tag{30}$$

Plugging (7) and (8) into (30), we can solve for optimal human capital investment $(e_t^{\mathrm{i}})^*$ as:

$$(e_t^{\mathrm{i}})^* = \left[\frac{\gamma(1-\alpha)}{\alpha}\bar{g}A_t^{\mathrm{i}}\tilde{k}_{t+1}\right]^{\frac{1}{1-\gamma}} \tag{31}$$

In (31), $\tilde{k}_{t+1}$ is the capital-effective human capital labour ratio. Using equations (6) and (21), we can write $\tilde{k}_{t+1}$ as:

$$\tilde{k}_{t+1} = \frac{k_{t+1}}{A_{t+1}} = \frac{k_{t+1}}{A_t[1 + g[(e_t^{\mathrm{i}})^*]]} = \frac{k_{t+1}}{A_t^{\mathrm{i}}\bar{g}[1 - \frac{1}{[(e_t^{\mathrm{i}})^*]^\gamma}]}$$

So we have

$$\tilde{k}_{t+1}A_t^{\mathrm{i}}\bar{g} = \frac{k_{t+1}}{1 - \frac{1}{[(e_t^{\mathrm{i}})^*]^\gamma}}$$

Now plug the above expression of $\tilde{k}_{t+1}A_t^{\mathrm{i}}\bar{g}$ into (31) ans we can write k_{t+1} in terms of $(e_t^{\mathrm{i}})^*$ as:

$$k_{t+1} = [(e_t^{\mathrm{i}})^*]^{1-\gamma}\left[1 - \frac{1}{[(e_t^{\mathrm{i}})^*]^\gamma}\right]\frac{\alpha}{\gamma(1-\alpha)} \tag{32}$$

(32) determines a one-to-one monotonic mapping from $(e_t^{\mathrm{i}})^*$ to k_{t+1}. And k_{t+1} is increasing in $(e_t^{\mathrm{i}})^*$. Then we can derive $(e_t^{\mathrm{i}})^*$ as an inverse function of k_{t+1} from (32) as $(e_t^{\mathrm{i}})^* = (e_t^{\mathrm{i}})^*(k_{t+1})$. We denote such $(e_t^{\mathrm{i}})^*$ as $e_t^{(3)}$. Then we have:

$$(e_t^{\mathrm{i}})^* = (e_t^{\mathrm{i}})^*(k_{t+1}) \equiv e_t^{(3)}(k_{t+1}) \tag{33}$$

As the growth rate of stock of knowledge of individual i, g_{t+1}^{i}, is a function of e_t^{i} (i.e. $g_{t+1} = g(e_t^{\mathrm{i}})$), combine (33) with (21) and we can see that g_{t+1}^{i} is also identical across individual skilled workers and can be denoted as $g_{t+1}^{\mathrm{i}} = g_{t+1}$. Because the stock of knowledge per skilled worker A_{t+1}^{i} starts at the level of unity, identical growth rate of the knowledge implies that stock of knowledge is also identical across individual skilled workers in every period. So we have $A_{t+1}^{\mathrm{i}} = A_{t+1}$.

From (25), (28) and (33) we can see that optimal amount of human capital investment is always identical across individual skilled workers, thus we have $(e_t^{\mathrm{i}})^* = e_t$. And the efficient labour supplied by each skilled worker h_{t+1}^{i} is identical. So we have: $h_{t+1}^{\mathrm{i}} = h_{t+1}$. We can summarize the formulation of human capital investment as:

$$e_t = \begin{cases} \underline{e} & \text{given} \quad e_t \leq 1 \\ e^{(2)}(k_{t+1}) & \text{given} \quad 1 < e_t < (\frac{\bar{g}}{\bar{g}-1})^{\frac{1}{\gamma}} \\ e_t^{(3)}(k_{t+1}) & \text{given} \quad e_t \geq (\frac{\bar{g}}{\bar{g}-1})^{\frac{1}{\gamma}} \end{cases} \tag{34}$$

(34) shows that the human capital investment takes three different patterns. And which pattern the human capital investment takes depends on the level of k_{t+1}, the capital-human capital ratio.

To see this, we first make the following assumption regarding parameters $\bar{g}$ and γ:

Assumption 2. *Parameters $\bar{g}$ and γ are set so that*

$$\left(\frac{\bar{g}}{\bar{g}-1}\right)^{\frac{1-\gamma}{\gamma}} > \bar{g}$$

Now we discuss how the human capital shift across various forms. $e_t = \underline{e}$ implies $e_t \leq 1$. And when $e_t = e^{(2)}(k_{t+1})$ holds, $e_t > 1$ has to hold. According to the formulation of $e^{(2)}(k_{t+1})$ in equation (28), $e_t > 1$ implies

$$\left[\frac{\gamma(1-\alpha)}{\alpha}\right]^{\frac{1}{1-\gamma}} > 1$$

So k_{t+1} satisfies:

$$k_{t+1} > \frac{\alpha}{\gamma(1-\alpha)} \equiv \underline{k} \tag{35}$$

Given $\underline{k}$ as defined in (35), we can show that the human capital investment $e_t = e^{(2)}(k_{t+1})$ holds only when $k_{t+1} > \underline{k}$. If $k_{t+1} \leq \underline{k}$, we will have $e_t = \underline{e}$. So the shift from $\underline{e}$ to $e^{(2)}(k_{t+1})$ requires k_{t+1} to grow beyond $\underline{k}$.

For the human capital investment to shift from $e^{(2)}(k_{t+1})$ to $e^{(3)}(k_{t+1})$, $e_t \geq [\bar{g}/(\bar{g}-1)]^{1/\gamma}$ has to hold. We know that $e^{(3)}(k_{t+1})$ is derived from equation (32). Using (32), we can see that when $e_t = [\bar{g}/(\bar{g}-1)]^{1/\gamma}$, k_{t+1} satisfies:

$$k_{t+1} = \left(\frac{\bar{g}}{\bar{g}-1}\right)^{\frac{1-\gamma}{\gamma}} \frac{1}{\bar{g}} \frac{\alpha}{\gamma(1-\alpha)} \equiv \bar{k} \tag{36}$$

Equation (32) shows that an increase in k_{t+1} leads to a higher $e^{(3)}(k_{t+1})$. Then $k_{t+1} \geq \bar{k}$ has to hold to maintain $e_t \geq [\bar{g}/(\bar{g}-1)]^{1/\gamma}$. On the other hand, according to equations (35), (36) as well as assumption 2, we have $\bar{k} > \underline{k}$. In this way, we can see that when the physical capital-human capital ratio k_{t+1} is beyond $\underline{k}$ but below $\bar{k}$, the human capital investment takes the form of $e^{(2)}(k_{t+1})$. If it goes beyond $\bar{k}$, it takes the form of $e^{(3)}(k_{t+1})$.

In this way, we can rewrite (34) to capture the interrelation between the human capital investment per person e_t^{i} and physical capital-human capital ratio k_{t+1} as:

$$e_t = \begin{cases} \underline{e} & \text{given} \quad k_{t+1} \leq \underline{k} \\ e^{(2)}(k_{t+1}) & \text{given} \quad \underline{k} < k_{t+1} < \bar{k} \\ e_t^{(3)}(k_{t+1}) & \text{given} \quad k_{t+1} \geq \bar{k} \end{cases} \tag{37}$$

The human capital per skilled worker h_{t+1} satisfies:

$$h_{t+1} = \begin{cases} 1 & \text{given} \quad k_{t+1} \leq \underline{k} \\ \left[e^{(2)}(k_{t+1})\right]^{\gamma} \equiv h^{(2)}(k_{t+1}) & \text{given} \quad \underline{k} < k_{t+1} < \bar{k} \\ \left[e_t^{(3)}(k_{t+1})\right]^{\gamma} \equiv h^{(3)}(k_{t+1}) & \text{given} \quad k_{t+1} \geq \bar{k} \end{cases} \tag{38}$$

Because $e^{(2)}(k_{t+1})$ and $e^{(3)}(k_{t+1})$ are increasing in k_{t+1}, the $h^{(2)}(k_{t+1})$ and $h^{(3)}(k_{t+1})$ in (38) are also increasing in k_{t+1}. In this way, the human capital investment e_t and the human capital accumulated by each skilled worker h_{t+1} take three different patterns. Which pattern is taken depends on the level of the capital-human capital ratio k_{t+1}.

As will be shown in the subsequent analysis, these three different formation of human capital investment further generates different patterns of growth, thus dividing the process of development into three different regimes. The growth of physical capital-human capital ratio, k_{t+1}, generates the transition across the regimes.

2.3. **Labour and Capital-Human Capital Ratio.** We now analyze the dynamics of capital-human capital ratio. Capital-human capital ratio is related to the supply of labour, the aggregate physical capital stock and the human capital per skilled worker. We begin with analyzing the labour supply, then we derive the aggregate physical capital stock and calculate the capital-human capital ratio.

2.3.1. *Labour Supply.* Suppose at the beginning, the population that is working is L and each individual in the working population gives birth to only one child in each of the subsequent periods. Therefore, the population that is working in every period is L. So in period $t+1$ (the period of adulthood for individuals born in t), the supply of skilled labour S_{t+1} and that of unskilled labour U_{t+1} satisfy:

$$L = S_{t+1} + U_{t+1} \tag{39}$$

The equilibrium supply of skilled and unskilled labour is such that each individual gains the same lifetime utility regardless of working as skilled labour or unskilled labour. Because an individuals' lifetime utility is increasing in his or her second period wealth, the equilibrium labour supply should generate identical second period wealth for all individuals. That is, the second period wealth individual i gains from working as a skilled worker equals to that from working as an unskilled worker. And according to equation (14), this implies that

$$I_{t+1}^{\mathrm{i,S}} = I_{t+1}^{\mathrm{i,U}} \tag{40}$$

holds for all $i (i = 1, 2, ...L)$.

Plug the formulation of $I_{t+1}^{\mathrm{i,S}}$ and $I_{t+1}^{\mathrm{i,U}}$ in (14) into (40) and use the property of identical human capital investment per skilled worker (i.e. $e_t^{\mathrm{i}} = e_t$), we can rewrite (40) as:

$$\tilde{w}_t A_t h_t + (b_t^{\mathrm{i}} - e_t) r_{t+1} = w_{t+1}^{\mathrm{U}} + b_t^i r_{t+1}$$

Based on (8), (11) and (12), we can rewrite the above equation as:

$$(1-\alpha) k_{t+1}^{\alpha} A_{t+1}^{1-\alpha} h_{t+1} - e_t \alpha A_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1} = (1-\alpha) \bar{X}^{\alpha} (A_{t+1}^{\mathrm{U}})^{1-\alpha} U_{t+1}^{-\alpha} \tag{41}$$

As can be seen from (37), e_t is either a constant or an increasing function of k_{t+1}. Because of this, h_{t+1} is either constant or increasing in k_{t+1}. Then (41) implies that the equilibrium supply of unskilled labour U_{t+1} is determined by A_{t+1}, A_{t+1}^{U}, and k_{t+1}. So U_{t+1} can be written as $U_{t+1} = U(A_{t+1}, A_{t+1}^{\mathrm{U}}, k_{t+1})$. From (41) we can solve for $U_{t+1} = U(A_{t+1}, A_{t+1}^{\mathrm{U}}, k_{t+1})$ as:

$$U_{t+1} = U(A_{t+1}, A_{t+1}^{\mathrm{U}}, k_{t+1}) = \left[\frac{(1-\alpha) \bar{X}^{\alpha} (A_{t+1}^{\mathrm{U}})^{1-\alpha}}{(1-\alpha) k_{t+1}^{\alpha} A_{t+1}^{1-\alpha} h_{t+1} - e_t \alpha A_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1}} \right]^{\frac{1}{\alpha}} \tag{42}$$

And based on equation (39), we can derive optimal supply of skilled labour S_{t+1} as:

$$S_{t+1} = L - U_{t+1} = L - U(A_{t+1}, A_{t+1}^{\mathrm{U}}, k_{t+1}) \tag{43}$$

Equation (43) shows that S_{t+1} is also determined by A_{t+1}, A_{t+1}^{U} and k_{t+1}.

2.3.2. *Capital-Human Capital Ratio at Equilibrium.* At equilibrium, the aggregate output in any given period t, Y_t, equals to $\sum_j I_t^{\mathrm{j}}$, the summary of the wealth of each adult individual

j in period t. As is shown in equation (18), the bequest left to the generation born in period t by individual j is βI_t^{j}. Therefore the total bequest in period t satisfies $\sum_j \beta I_t^j = \beta Y_t$.

In period $t+1$, the supply of skilled labour is S_{t+1}, as formulated in equation (43). And each individual that works as skilled labour in period $t+1$ invests e_t into human capital accumulation in period t. So the total educational expenditure is $S_{t+1}e_t$. Based on the assumption that capital fully depreciates at the end of each period, the aggregate capital stock in period $t+1$, K_{t+1}, is the net savings in period t. And it can be calculated by subtracting the total bequest in t with the total educational expenditure:

$$K_{t+1} = \beta Y_t - S_{t+1}e_t \tag{44}$$

Now we can derive capital-skilled labour ratio k_{t+1}. The formulation of k_{t+1} in equation (5) indicates that aggregate capital stock K_{t+1} can be written as: $K_{t+1} = k_{t+1}S_{t+1}h_{t+1}$. Using equation (43), we can write K_{t+1} as:

$$K_{t+1} = k_{t+1}S_{t+1}h_{t+1} = k_{t+1}[L - U(A_{t+1}, A_{t+1}^{\mathrm{U}}, k_{t+1})]h_{t+1}$$

We can also write Y_t in terms of capital-human capital ratio k_t as

$$\begin{aligned} Y_t &= Y_t^{\mathrm{S}} + Y_t^{\mathrm{U}} = k_t^{\alpha} S_t h_t A_t^{1-\alpha} + \bar{X}^{\alpha}(A_t^{\mathrm{U}} U_t)^{1-\alpha} \\ &= [L - U(A_t, A_t^{\mathrm{U}}, k_t)]k_t^{\alpha} A_t^{1-\alpha} h_t + \beta \bar{X}^{\alpha}(A_t^{\mathrm{U}})^{1-\alpha} U(A_t, A_t^{\mathrm{U}}, k_t)^{1-\alpha} \end{aligned}$$

So equation (44) can be written as:

$$\begin{aligned} &K_{t+1} = k_{t+1}[L - U(A_{t+1}, A_{t+1}^{\mathrm{U}}, k_{t+1})]h_{t+1} = \beta Y_t - e_t S_{t+1} \\ &= \beta[L - U(A_t, A_t^{\mathrm{U}}, k_t)]k_t^{\alpha} A_t^{1-\alpha} h_t + \beta \bar{X}^{\alpha}(A_t^{\mathrm{U}})^{1-\alpha} U(A_t, A_t^{\mathrm{U}}, k_t)^{1-\alpha} - e_t[L - U(A_{t+1}, A_{t+1}^{\mathrm{U}}, k_{t+1})] \end{aligned}$$

Move the term $e_t[L - U(A_{t+1}, A_{t+1}^{\mathrm{U}}, k_{t+1})]$ to the left hand side of the equation and we have:

$$\begin{aligned} &[L - U(A_{t+1}, A_{t+1}^{\mathrm{U}}, k_{t+1})](k_{t+1}h_{t+1} + e_t) = \beta[L - U(A_t, A_t^{\mathrm{U}}, k_t)]k_t^{\alpha} A_t^{1-\alpha} h_t \\ &+ \beta[\bar{X}^{\alpha}(A_t^{\mathrm{U}})^{1-\alpha} U(A_t, A_t^{\mathrm{U}}, k_t)^{-\alpha}]U(A_t, A_t^{\mathrm{U}}, k_t) \end{aligned} \tag{45}$$

Note that we can solve for $\bar{X}^{\alpha}(A_t^{\mathrm{U}})^{1-\alpha} U(A_t, A_t^{\mathrm{U}}, k_t)^{-\alpha}$ from equation (41) in terms of k_t as:

$$\bar{X}^{\alpha}(A_t^{\mathrm{U}})^{1-\alpha} U(A_t, A_t^{\mathrm{U}}, k_t)^{-\alpha} = k_t^{\alpha} A_t^{1-\alpha} h_t - \frac{\alpha e_{t-1} k_t^{\alpha-1} A_t^{1-\alpha}}{1-\alpha} \tag{46}$$

Plug equation (46) into equation (45) and we have:

$$[L - U(A_{t+1}, A_{t+1}^{\mathrm{U}}, k_{t+1})](k_{t+1}h_{t+1} + e_t) = \beta L k_t^{\alpha} A_t^{1-\alpha} h_t - U(A_t, A_t^{\mathrm{U}}, k_t)\frac{\alpha e_{t-1} \beta k_t^{\alpha-1} A_t^{1-\alpha}}{1-\alpha} \tag{47}$$

Equation (47) shows that k_{t+1} is implicitly determined by k_t, A_t, A_t^{U}, A_{t+1} and A_{t+1}^{U}. Though we can not solve for k_{t+1} explicitly, we can make analysis of the dynamics of k_{t+1} on the basis of equation (47).

Note that equation (42) shows that for given A_{t+1}, A_{t+1}^{U}, $U(A_{t+1}, A_{t+1}^{\mathrm{U}}, k_{t+1})$ is decreasing with respect to k_{t+1}. So $-U(A_{t+1}, A_{t+1}^{\mathrm{U}}, k_{t+1})$ is increasing in k_{t+1} for given A_{t+1} and A_{t+1}^{U}.

And the formulation of h_{t+1} in (20) and e_t in (34) show that e_t (hence h_{t+1}) is a nondecreasing function of k_{t+1}. Then the left hand side of equation (47), $[L-U(A_{t+1}, A^{\mathrm{U}}_{t+1}, k_{t+1})](k_{t+1}h_{t+1}+e_t)$, is an increasing function of k_{t+1} given A_{t+1} and A^{U}_{t+1}. So if $k_{t+1} > k_t$, the following holds for any given A_{t+1} and A^{U}_{t+1}:

$$[L - U(A_{t+1}, A^{\mathrm{U}}_{t+1}, k_{t+1})](k_{t+1}h_{t+1} + e_t) - [L - U(A_{t+1}, A^{\mathrm{U}}_{t+1}, k_t)](k_t h_t + e_{t-1}) > 0 \quad (48)$$

Note that (34) and (20)show that e_{t-1} and h_t are functions of k_t. And (47) shows that $L - U(A_{t+1}, A^{\mathrm{U}}_{t+1}, k_{t+1})](k_{t+1}h_{t+1} + e_t)$ is a function of k_t. So given A_{t+1} and A^{U}_{t+1}, the left hand side of (48) is a function of k_t and can be defined as $\tau(k_t)$. And $\tau(k_t)$ can be formulated as:

Definition 2. *The left hand side of* (48) *is defined as* $\tau(k_t)$ *and* $\tau(k_t)$ *can be written as:*

$$\begin{aligned}\tau(k_t) &= [L - U(A_{t+1}, A^{U}_{t+1}, k_{t+1})](k_{t+1}h_{t+1} + e_t) - [L - U(A_{t+1}, A^{U}_{t+1}, k_t)](k_t h_t + e_{t-1}) \\ &= L(\beta k_t^{\alpha} A_t^{1-\alpha} h_t - k_t h_t - e_{t-1}) + U(A_{t+1}, A^{U}_{t+1}, k_t)(k_t h_t + e_{t-1}) - U(A_t, A^{U}_t, k_t)\frac{\alpha e_{t-1}\beta k_t^{\alpha-1} A_t^{1-\alpha}}{1-\alpha}\end{aligned} \quad (49)$$

In (49), the second equality follows from rewriting $[L - U(A_{t+1}, A^{\mathrm{U}}_{t+1}, k_{t+1})](k_{t+1} + e_t)$ in terms of k_t based on equation (47).

According to (48) and the definition of $\tau(k_t)$ defined by (49), we can see that $\tau(k_t) > 0$ implies $k_{t+1} > k_t$, or equivalently, k_t is growing. Vice versa, $\tau(k_t) < 0$ implies that k_t is diminishing. In this way, $\tau(k_t)$, as formulated in (49), is the fundamental reference to the subsequent analysis of the dynamics of k_{t+1}.

2.4. **Technological Change.** We now turn to the last but not the least part of the model: technology. Technological change takes place in two lines: the other is featured with the growth of the productivity of skilled labour, A_{t+1}, and the other is featured with the growth of the productivity of unskilled labour, A^{U}_{t+1}. The growth of A^{U}_{t+1} is assumed as follows:

Assumption 3. *Denote the growth rate of* A^{U}_{t+1} *as* g^{U}_{t+1}. g^{U}_{t+1} *is determined by the supply of unskilled labour in the previous period* t, U_t:

$$g^{U}_{t+1} = g(U_t) \quad (0 \le U_t \le L) \quad (50)$$

$g(U_t)$ *in* (50) *satisfies:* $g(0) = 0$, $g'(0) > 0$, $g''(U_t) < 0$.

Assumption 3 shows that the supply of unskilled labour augments the growth of its productivity in a diminishing manner. This is similar to the formulation of technological change in Galor and Weil (2000). And similar to Galor and Weil (2000), assumption 3 guarantees that there is technological change even when the economy is in the primitive stage of development.

The formulation of the growth rate of A^{U}_{t+1}, g^{U}_{t+1}, in assumption 3 implies that for $0 \leq U_t \leq L$, there is a maximum level of g^{U}_{t+1}, which can be denoted as $\max_{0 \leq U_t \leq L}\{g^{\mathrm{U}}_{t+1}\}$. On the other hand, (21) indicates that the growth rate of the knowledge stock per skilled worker, g_{t+1}, is related to the parameter $\bar{g}$. The relation between $\bar{g}$ and $\max_{0 \leq U_t \leq L}\{g^{\mathrm{U}}_{t+1}\}$ is assumed as follows:

Assumption 4. *The parameter $\bar{g}$ in (21) and $max_{0 \leq U_t \leq L}\{g^{U}_{t+1}\}$ satisfy:*

$$\bar{g} - 1 > \frac{1}{1-\alpha}[max_{0 \leq U_t \leq L}\{g^{U}_{t+1}\}] \tag{51}$$

In (51), $1/(1-\alpha) > 1$ holds. And note that (21), which formulates g_{t+1}, indicates that $\lim_{e_t \to +\infty} g_{t+1} = \bar{g} - 1$. This implies that

$$g_{t+1} > \frac{1}{1-\alpha}[\max_{0 \leq U_t \leq L}\{g^{\mathrm{U}}_{t+1}\}] \tag{52}$$

holds for sufficiently large e_t.

Equation (52) indicates that when the human capital investment is large enough, the productivity of human capital grows faster than that of the productivity of unskilled labour by several times. As the productivity of the human capital reflects the productivity of "industry" and that of unskilled labour reflects the productivity of "agriculture", equation (52) (together with assumption 4) indicates that as a result of increasing human capital investment over time, the growth of the productivity of the industry outperforms that of the agriculture. This captures the co-existence of ever-increasing human capital investment and massive productivity growth in industry relative to agriculture, which is the prominent feature of the advanced stage of development.

3. Three Regimes of Growth and the Transition in between

The process of development consists of three different regimes of growth. This is because the human capital investment e_t takes three different patterns (see (37)) and each pattern of the human capital investment yields a distinctive pattern of growth, thus generating a specific regime of growth.

The capital-human capital ratio starts at a level which sufficiently low (i.e.$k_{t+1} \leq \underline{k}$). According to (37), the human capital investment is trapped at a low level $\underline{e}$, restricting the human capital acquired by each skilled worker to 1. This can be seen as a "quasi-Malthusian trap" of human capital investment, as is mentioned in the introduction. The human capital investment is too low to induce the growth of the productivity of skilled labour ($e_t = \underline{e} < (\frac{\bar{g}}{\bar{g}-1})^{1/\gamma}$), technological change consists of the growing productivity of unskilled labour only, which is powered by the supply of unskilled labour. Technological change as such is usually slow.[10] The capital-human capital ratio is also growing and

[10]This means that the growth rate of productivity of unskilled labour A^{U}_{t+1} which is powered by supply of unskilled labour (see equation(50)) is set to be low. This captures the slow technological change in pre-modern times.

its growth is driven by growing productivity of unskilled labour. In general, this regime is featured with slow technological change as well as low and fixed human capital investment and human capital accumulation. It roughly corresponds to the late medieval period in history. We thus refer to this regime as the Late Medieval Regime , which ranges from 1300 to 1600.

As the capital-human capital ratio is growing, it becomes higher but still not high enough (i.e. $\underline{k} < k_{t+1} < \bar{k}$), which gives rise to a different pattern of human capital investment ($e_t = e^{(2)}(k_{t+1})$). This results in a different pattern of the human capital per skilled worker($h_{t+1} = h^{(2)}(k_{t+1})$). The human capital investment leaves the "quasi-Malthusian trap" and starts growing mildly along with increasing capital-human capital ratio. Correspondingly, the human capital accumulated by each skilled worker goes beyond unity and increases with growing capital-human capital ratio. Because the capital-human capital ratio is not high enough, the human capital investment is not large enough to trigger the growth of productivity of skilled labour ($e_t < (\frac{\bar{g}}{\bar{g}-1})^{1/\gamma}$ still holds). Technological change follows the same pattern as in the previous regime, which consists of the growth of the productivity of unskilled labour only. Similarly, the growth of the capital-human capital ratio is driven by the growing productivity of unskilled labour. This regime, which features with slow technological change but mildly growing human capital investment, roughly corresponds to the early modern period in history. We thus refer to it as the Early Modern Regime, which ranges from 1600 to 1800.

As the capital-human capital ratio becomes sufficiently high (i.e. $k \geq \bar{k}$), it gives rise to the human capital investment that is large enough to augment the growth of productivity of skilled labour (i.e. $e_t \geq (\frac{\bar{g}}{\bar{g}-1})^{1\gamma}$). Similar to the previous regime, the human capital investment, which takes the form $e_t = e^{(3)}(k_{t+1})$, increases along with growing capital-human capital ratio. Technological change now takes a "modern fashion", which sees growing productivity of both skilled labour and unskilled labour. And the capital-human capital ratio continues to grow, which is driven by the growing productivity of skilled labour and that of unskilled labour. The sustainable growth of the capital-human capital ratio in turn raises the growth rate of the productivity of skilled labour and leads it to converge to a high level. This generates the "sustainable growth" similar to its counterpart in canonical long-run growth literature such as Galor and Weil (2000). We can then refer to this regime as the Modern Growth Regime, which ranges from 1800 to 1914.

As is discussed before, which regime the economy is in depends on the pattern of human capital investment. And the human capital investment is further determined by the level of the capital-human capital ratio. So the growth of capital-human capital ratio from a low level to a sufficiently high level is the key to transition from one regime to another. The growth of the capital-human capital ratio takes place in all the regimes and is augmented either by the growing productivity of unskilled labour only or by the growing productivity of both skilled and unskilled labour. To ensure the growth of capital-human capital ratio in the Late Medieval Regime (1300-1600) and the Early modern Regime (1600-1800), we make the following assumption regarding relevant parameters:

Assumption 5. *Parameters α, β, γ and $\underline{e}$ are set to satisfy:*

$$\frac{\underline{e}\alpha}{1-\alpha}+\underline{e}-\beta\left(\frac{\underline{e}\alpha}{1-\alpha}\right)^{\alpha}>0 \tag{53}$$

$$\frac{\alpha}{\gamma(1-\alpha)}+1-\left(\frac{\alpha}{1-\alpha}\right)^{\alpha}\gamma^{1-\alpha}\beta>0 \tag{54}$$

(53) and (54) guarantee the growth of k_{t+1} in late medieval regime (1300-1600) and the early modern regime (1600-1800). This enables k_{t+1} to grow throughout the pre-modern period and drive the economy towards modernity. In the next section we will show that as k_{t+1} grows throughout the period from 1300 to 1914, the skill premium evolves accordingly and exhibits the pattern as observed in Figure I.

4. Skill Premium in the Three Regimes of Growth

This section analyses the evolution of skill premium in the three regimes of growth outline before. To do this, we first derive the skill premium by calculating $w^{\mathrm{S}}_{t+1}/w^{\mathrm{U}}_{t+1}$, the ratio of the wage of skilled labour to that of unskilled labour.

(12) shows that the wage of unskilled labour satisfies: $w^{\mathrm{U}}_{t+1}=(1-\alpha)\bar{X}^{\alpha}(A^{\mathrm{U}}_{t})^{1-\alpha}U_t^{-\alpha}$. Then using equation (41) we can write w^{U}_{t+1} as:

$$w^{\mathrm{U}}_{t+1}=(1-\alpha)\bar{X}^{\alpha}(A^{\mathrm{U}}_{t+1})^{1-\alpha}U_{t+1}^{-\alpha}=(1-\alpha)k_{t+1}^{\alpha}A_{t+1}^{1-\alpha}h_{t+1}-e_t\alpha A_{t+1}^{1-\alpha}k_{t+1}^{\alpha-1}$$

Combine equation (10) with the equation above, we can write $w^{\mathrm{S}}_{t+1}/w^{\mathrm{U}}_{t+1}$ as:

$$\frac{w^{S}_{t+1}}{w^{U}_{t+1}}=\frac{(1-\alpha)k_{t+1}^{\alpha}A_{t+1}^{1-\alpha}h_{t+1}}{(1-\alpha)k_{t+1}^{\alpha}A_{t+1}^{1-\alpha}h_{t+1}-e_t\alpha A_{t+1}^{1-\alpha}k_{t+1}^{\alpha-1}}=\frac{1}{1-\frac{\alpha}{1-\alpha}\frac{e_t}{k_{t+1}h_{t+1}}} \tag{55}$$

Equation (55) shows that human capital investment, human capital per skilled worker and capital-human capital ratio jointly determine the level of skill premium. Because capital investment e_t and the human capital per skilled worker h_{t+1} are either constant or functions of k_{t+1}, as (37) and (38) show, the skill premium is then determined by the capital-human capital ratio k_{t+1}. How the growth of k_{t+1} influences the evolution of the skill premium will be analysed in the following.

4.1. **Skill Premium in the Late Medieval Regime (1300-1600).** This is a regime of stagnation featured with low level of the capital-human capital ratio (i.e. $k_{t+1}<\underline{k}$). The human capital investment e_t is trapped at the low and fixed level of $\underline{e}$: $e_t=\underline{e}$, which restricts the human capital accumulated by each skilled worker to 1. Because the human capital investment is too low, there is no growth of the productivity of skilled labour, A_{t+1}. A_{t+1} is constant and can be set to 1. Technological change consists of the growth of A^{U}_{t+1}, the productivity of unskilled labour, only. The growth of A^{U}_{t+1} is powered by U_t, as formulated in equation (50).

As for the skill premium, the following proposition holds:

Proposition 1. *Skill premium declines as the capital-human capital ratio k_{t+1} grows*

Proof. Because $e_t = \underline{e}$, equation (55) can be written as:

$$\frac{w^{\mathrm{S}}_{t+1}}{w^{\mathrm{U}}_{t+1}} = \frac{1}{1 - \frac{\alpha \underline{e}}{(1-\alpha)k_{t+1}}}$$

So we have

$$\frac{\partial}{\partial k_{t+1}} \frac{w^{\mathrm{S}}_{t+1}}{w^{\mathrm{U}}_{t+1}} = -\frac{1}{\left(1 - \frac{\alpha \underline{e}}{(1-\alpha)k_{t+1}}\right)^2} \frac{\alpha \underline{e}}{(1-\alpha)k^2_{t+1}} < 0$$

Therefore the skill premium is decreasing with respect to k_{t+1}. As k_{t+1} grows, skill premium declines. □

Proposition 1 shows that growing k_{t+1} will result in declining skill premium. This indicates that declining skill premium as depicted in Figure 1 may be the result of growing k_{t+1}. We now turn to the analysis of the growth of k_{t+1}.

We first derive the supply of unskilled labour. With $A_{t+1} = 1$, supply of unskilled labour U_{t+1}, which is a function of A_{t+1}, A^{U}_{t+1} and k_{t+1}, can be written as: $U_{t+1} = U^{(1)}(A^{\mathrm{U}}_{t+1}, k_{t+1})$. Furthermore, the human capital investment e_t is restrained at $\underline{e}$ and human capital per skilled worker h_{t+1} is unity. Then based on equation (42), we can write the optimal supply of unskilled labour as:

$$U_{t+1} = U^{(1)}(A^{\mathrm{U}}_{t+1}, k_{t+1}) = \left[\frac{(1-\alpha)\bar{X}^{\alpha}(A^{\mathrm{U}}_{t+1})^{1-\alpha}}{(1-\alpha)k^{\alpha}_{t+1} - \underline{e}\alpha k^{\alpha-1}_{t+1}}\right]^{\frac{1}{\alpha}} \tag{56}$$

Based on equation (56), the optimal supply of skilled labour can be written as: $S_{t+1} = L - U^{(1)}(A^{\mathrm{U}}_{t+1}, k_{t+1})$. Equation (56) shows that $U^{(1)}(A^{\mathrm{U}}_{t+1}, k_{t+1})$ is increasing in A^{U}_{t+1} for given k_{t+1}. It also shows that $(1-\alpha)k^{\alpha}_{t+1} - \underline{e}\alpha k^{\alpha-1}_{t+1} > 0$ has to hold to maintain positive supply of unskilled labour. This implies that

$$k_{t+1} > \frac{\underline{e}\alpha}{1-\alpha}$$

must hold. Therefore even when the capital-human capital ratio is low, it has to be above the lower bound $\underline{e}\alpha/(1-\alpha)$.

As mentioned before, a general characterization of the dynamics of k_{t+1} is the formulation of $\tau(k_t)$ in (49). We have $e_t = \underline{e}$, $h_t = 1$ and $A_t = 1$. And $U^{(1)}(A^{\mathrm{U}}_{t+1}, k_{t+1})$ is increasing in

A^{U}_{t+1}. We then specify $\tau(k_t)$ as $\tau(k_t)^{(1)}$. And $\tau(k_t)^{(1)}$ can be written as:

$$\begin{aligned}\tau^{(1)}(k_t) &= L(\beta k_t^{\alpha} - k_t - \underline{e}) + U^{(1)}(A^U_{t+1}, k_t)(k_t + \underline{e}) - U^{(1)}(A^U_t, k_t)\frac{\alpha\underline{e}\beta k_t^{\alpha-1}}{1-\alpha} \\ &> L(\beta k_t^{\alpha} - k_t - \underline{e}) + U^{(1)}(A^U_t, k_t)\left(k_t + \underline{e} - \frac{\alpha\underline{e}\beta k_t^{\alpha-1}}{1-\alpha}\right) \\ &= L(\beta k_t^{\alpha} - k_t - \underline{e}) + \left[\frac{(1-\alpha)\bar{X}^{\alpha}(A^{\mathrm{U}}_t)^{1-\alpha}}{(1-\alpha)k_t^{\alpha} - \underline{e}\alpha k_t^{\alpha-1}}\right]^{\frac{1}{\alpha}}\left(k_t + \underline{e} - \frac{\alpha\underline{e}\beta k_t^{\alpha-1}}{1-\alpha}\right) \equiv \psi^{(1)}(A^{\mathrm{U}}_t, k_t) \qquad (57)\end{aligned}$$

From (57) we can see that $\psi^{(1)}(A^{\mathrm{U}}_t, k_t) > 0$ results in $\tau(k_t)^{(1)} > 0$, which implies that $\psi^{(1)}(A^{\mathrm{U}}_t, k_t) > 0$ is the sufficient condition for $k_{t+1} > k_t$. The subsequent analysis will focus on the sign of $\psi^{(1)}(A^{\mathrm{U}}_t, k_t)$.

Equation (53) in assumption 5 guarantees that $k_t + \underline{e} - \frac{\alpha\underline{e}\beta k_t^{\alpha-1}}{1-\alpha} > 0$ holds when $k_t = \underline{e}\alpha/(1-\alpha)$. Because it is increasing in k_t, $k_t + \underline{e} - \frac{\alpha\underline{e}\beta k_t^{\alpha-1}}{1-\alpha} > 0$ holds for all $k_t > \underline{e}\alpha/(1-\alpha)$. Moreover, we can see that given $k_t \to \underline{e}\alpha/(1-\alpha)$, have$(1-\alpha)k_t^{\alpha} - \underline{e}\alpha k_t^{\alpha-1} \to 0$ holds. And when $k_t \to \underline{e}\alpha/(1-\alpha)$, $\beta k_t^{\alpha} - k_t - \underline{e}$ is finite. Therefore given A^{U}_t, we have:

$$\lim_{k_t\to\underline{e}\alpha/(1-\alpha)}\psi^{(1)}(A^{\mathrm{U}}_t, k_t) = +\infty$$

We can also verify that

$$\lim_{k_t\to+\infty}\psi^{(1)}(A^{\mathrm{U}}_t, k_t) = -\infty$$

hold for given A^{U}_t. Thus there exists $k^*_{1,t} = k_1(A^{\mathrm{U}}_t)$ and $\eta_{1,t} > 0$ such that

$$\begin{aligned}&\psi^{(1)}(A^{\mathrm{U}}_t, k^*_{1,t}) = 0 \\ &\psi^{(1)}(A^{\mathrm{U}}_t, k_t) > 0 \quad \underline{e}\alpha/(1-\alpha) < k_t < k^*_{1,t} \\ &\psi^{(1)}(A^{\mathrm{U}}_t, k_t) < 0 \quad k^*_{1,t} < k_t < k^*_{1,t} + \eta_{1,t}\end{aligned}$$

In this way, $k^*_{1,t}$ is the least value to maintain $\psi^{(1)}(A^{\mathrm{U}}_t, k^*_{1,t}) \le 0$ given $k_t > \underline{e}\alpha/(1-\alpha)$. And we can formulate $k^*_{1,t}$ as:

$$k^*_{1,t} = \inf\{k_t > \underline{e}\alpha/(1-\alpha) | \psi^{(1)}(A^{\mathrm{U}}_t, k_t) \le 0\} \qquad (58)$$

With $k^*_{1,t}$ formulated in (58), we have the following proposition characterizing the dynamics of k_t in the late medieval era:

Proposition 2. *If the initial value of k_t satisfies $\underline{e}\alpha/(1-\alpha) < k_t \le \alpha/\gamma(1-\alpha)^2$, the growth of A^U_t will raise k_t above $\underline{k}$, the threshold that distinguishes the late medieval regime and the early modern regime.*

Proof. As discussed before, $k_t + \underline{e} - \frac{\alpha\underline{e}\beta k_t^{\alpha-1}}{1-\alpha} > 0$ holds for all $k_t > \underline{e}\alpha/(1-\alpha), \underline{k}$. Also from equation (56), we can see that $(1-\alpha)k_t^{\alpha} - \underline{e}\alpha k_t^{\alpha-1} > 0$ must hold for all $k_t > \underline{e}\alpha/(1-\alpha)$. Then from equation (57) we can derive the partial derivative of $\psi^{(1)}(A^{\mathrm{U}}_t, k_t)$ with respect to A^{U}_t as:

$$\frac{\partial\psi^{(1)}(A^{\mathrm{U}}_t, k_t)}{\partial A^{\mathrm{U}}_t} = \left[\frac{(1-\alpha)\bar{X}^{\alpha}}{(1-\alpha)k_t^{\alpha} - \underline{e}\alpha k_t^{\alpha-1}}\right]^{\frac{1}{\alpha}}\left(k_t + \underline{e} - \frac{\alpha\underline{e}\beta k_t^{\alpha-1}}{1-\alpha}\right)\frac{1-\alpha}{\alpha}(A^{\mathrm{U}}_t)^{\frac{1-\alpha}{\alpha}-1} > 0$$

So we have $\partial\psi^{(1)}(A_t^{\mathrm{U}}, k_t)/\partial A_t^{\mathrm{U}} > 0$. We also have $k_{1,t}^*$ as formulated in (58). Then according to theorem 1 in Milgrom and Roberts (1994), $k_{1,t}^*$ increases as A_t^{U} grows. Because the supply of unskilled labour generates a positive growth rate, there is continual growth of A_t^{U}. This leads $k_{1,t}^*$ to grow over time. And there exists $T_1 \geq 0$ such that $k_{1,t}^* > \underline{k}$ holds for $t \geq T_1$. In this way, for all $t \geq T_1$, $\psi(A_t^{\mathrm{U}}, k_t) > 0$ holds for all $\bar{e}\alpha/(1-\alpha) < k_t < \underline{k}$.

On the other hand, We know from equation (57) that $\tau^{(1)}(k_t) > \psi^{(1)}(A_t^{\mathrm{U}}, k_t)$ holds. Because $\psi(A_t^{\mathrm{U}}, k_t) > 0$ holds for all $t \geq T_1$, then $\tau^{(1)}(k_t) > \psi^{(1)}(A_t^{\mathrm{U}}, k_t) > 0$ must hold for all $t \geq T_1$. With $\tau^{(1)}(k_t) > 0$, we can conclude that k_t will grow continually given $t \geq T_1$. In this way, k_t will grow all the way towards the threshold level $\underline{k}$. □

Proposition 2 indicates that capital-skilled labour ratio, k_t, which starts below $\underline{k}$, the threshold level that distinguishes the late medieval regime and the early modern regime, will endogenously grow above this threshold, bringing the economy into the next regime. The growth of k_t is triggered by the growth of A_t^{U}.

Note that proposition 1 indicates that skill premium declines as capital-skilled labour ratio increases. And proposition 2 guarantees endogenously growing capital-human capital ratio. In this way, proposition 2 and proposition 1 indicate that the skill premium will exhibit a declining pattern as depicted in Figure 1.

4.2. **Skill Premium in the Early Modern Regime: 1600-1800.** In this regime, capital-human capital ratio becomes high (i.e. $k_{t+1} > \underline{k}$), which incurs a change in human capital investment. As a result, human capital investment is an increasing function of capital-human capital ratio (i.e. $e_t = e^{(2)}(k_{t+1})$ in (37)), which allows the human capital investment to grow as the capital-human capital ratio increases. In this way, the human capital accumulated by each skilled worker is also growing. On the other hand, the capital-human capital ratio is still not high enough (i.e. $k_{t+1} < \bar{k}$) to generate a sufficiently large human capital investment which augments the productivity of human capital. Technological change follows a similar fashion as the previous regime and consists of the growth of the productivity of unskilled labour, A_{t+1}^{U}, only. Generally speaking, this regime sees slow technological progress but mild growth of human capital investment.

Despite slow technological progress, the change in human capital investment does alter the pattern of the skill premium, as is shown in the following proposition:

Proposition 3. *Given $\underline{k} < k_{t+1} < \bar{k}$ and $e_t = e^{(2)}(k_{t+1})$ as formulated in (37), the skill premium will stay constant.*

Proof. We know from the previous discussion that when $k_{t+1} > \underline{k}$, the corresponding e_t will be greater than 1. So the human capital production function takes the form $h_{t+1} = e_t^{\gamma}$. So using equation (55), we can write the skill premium as:

$$\left(\frac{w_{t+1}^{\mathrm{S}}}{w_{t+1}^{\mathrm{U}}}\right)^{(2)} = \frac{1}{1 - \frac{\alpha e_t^{1-\gamma}}{(1-\alpha)k_{t+1}}} = \frac{1}{1 - \frac{\alpha[e^{(2)}(k_{t+1})]^{1-\gamma}}{(1-\alpha)k_{t+1}}}$$

Plugging the formulation of $e^{(2)}(k_{t+1})$ in equation (28) into the equation above, we can then write the skill premium as:

$$\left(\frac{w^{S}_{t+1}}{w^{U}_{t+1}}\right)^{(2)} = \frac{1}{1 - \frac{\alpha}{1-\alpha}\frac{\gamma(1-\alpha)k_{t+1}}{\alpha k_{t+1}}} = \frac{1}{1-\gamma} \tag{59}$$

As can be seen from (59), skill premium is determined by the exogenous parameter γ only. This implies that the skill premium stays constant. □

Proposition 3 not only captures the stable pattern of the skill premium from 1600 to 1800 as depicted in Figure 1, but also proposes a possible reason for such stability. From

While skill premium remains constant, the capital-human capital ratio k_{t+1} continues to grow. And the human capital investment e_t and human capital per skilled worker h_{t+1} satisfy:

$$e_t = e^{(2)}_t(k_{t+1}) = \left[\frac{\gamma(1-\alpha)}{\alpha}k_{t+1}\right]^{\frac{1}{1-\gamma}} \quad h_{t+1} = \left[\frac{\gamma(1-\alpha)}{\alpha}k_{t+1}\right]^{\frac{\gamma}{1-\gamma}} \tag{60}$$

As the human capital investment is still not large enough to spur the growth of the productivity of human capital, $A_{t+1} = 1$ still holds. Then the supply of unskilled labour, U_{t+1} is determined by A^{U}_{t+1} and k_{t+1} and can be denoted as $U_{t+1} = U^{(2)}(A^{U}_{t+1}, k_{t+1})$. Combine $A_{t+1} = 1$ and (60) with (42), we can derive the optimal supply of unskilled labour $U_{t+1} = U^{(2)}(A^{U}_{t+1}, k_{t+1})$ as:

$$U^{(2)}(A^{U}_{t+1}, k_{t+1}) = \left[\frac{\alpha}{\gamma(1-\alpha)}\right]^{\frac{\gamma}{\alpha(1-\gamma)}} \left[\frac{\bar{X}^{\alpha}(A^{U}_{t+1})^{1-\alpha}}{(1-\gamma)k_{t+1}^{\alpha+\frac{\gamma}{1-\gamma}}}\right]^{\frac{1}{\alpha}} \tag{61}$$

Denote the dynamics of k_{t+1} typical for the early modern regime as $\tau^{(2)}(k_t)$. Plugging (61), (60) and $A_{t+1} = 1$ into (49), we can then derive $\tau^{(2)}(k_t)$ as:

$$\begin{aligned}
\tau^{(2)}(k_t) = L &\left(\beta k_t^{\alpha}\left[\frac{\gamma(1-\alpha)}{\alpha}k_t\right]^{\frac{\gamma}{1-\gamma}} - k_t\left[\frac{\gamma(1-\alpha)}{\alpha}k_t\right]^{\frac{\gamma}{1-\gamma}} - \left[\frac{\gamma(1-\alpha)}{\alpha}k_t\right]^{\frac{1}{1-\gamma}}\right) \\
&+ U^{(2)}(A^{U}_{t+1}, k_t)\left(\left[\frac{\gamma(1-\alpha)}{\alpha}\right]^{\frac{\gamma}{1-\gamma}} k_t^{\frac{1}{1-\gamma}} + \left[\frac{\gamma(1-\alpha)}{\alpha}k_t\right]^{\frac{1}{1-\gamma}}\right) \\
&- U^{(2)}(A^{U}_{t}, k_t)\frac{\alpha\beta}{1-\alpha}\left[\frac{\gamma(1-\alpha)}{\alpha}k_t\right]^{\frac{1}{1-\gamma}} k_t^{\alpha-1}
\end{aligned}$$

From equation (61), we can see that if we take k_{t+1} as given, $U^{(2)}(A^{U}_{t+1}, k_{t+1}) > U^{(2)}(A^{U}_{t}, k_{t+1})$ holds when $A^{U}_{t+1} > A^{U}_{t}$. Therefore we can derive the following inequality:

$$
\begin{aligned}
&\tau^{(2)}(k_t) > L\left(\beta k_t^{\alpha}\left[\frac{\gamma(1-\alpha)}{\alpha}k_t\right]^{\frac{\gamma}{1-\gamma}} - k_t\left[\frac{\gamma(1-\alpha)}{\alpha}k_t\right]^{\frac{\gamma}{1-\gamma}} - \left[\frac{\gamma(1-\alpha)}{\alpha}k_t\right]^{\frac{1}{1-\gamma}}\right) \\
&+ U^{(2)}(A_t^{\mathrm{U}}, k_t)\left(\left[\frac{\gamma(1-\alpha)}{\alpha}\right]^{\frac{\gamma}{1-\gamma}} k_t^{\frac{1}{1-\gamma}} + \left[\frac{\gamma(1-\alpha)}{\alpha}k_t\right]^{\frac{1}{1-\gamma}} - \frac{\alpha\beta}{1-\alpha}\left[\frac{\gamma(1-\alpha)}{\alpha}\right]^{\frac{1}{1-\gamma}} k_t^{\alpha+\frac{\gamma}{1-\gamma}}\right) \\
&= L\left(\beta\left[\frac{\gamma(1-\alpha)}{\alpha}\right]^{\frac{\gamma}{1-\gamma}} k_t^{\alpha+\frac{\gamma}{1-\gamma}} - \left[\frac{\gamma(1-\alpha}{\alpha}\right]^{\frac{\gamma}{1-\gamma}} k_t^{\frac{1}{1-\gamma}} - \left[\frac{\gamma(1-\alpha)}{\alpha}k_t\right]^{\frac{1}{1-\gamma}}\right) + \\
&\left[\frac{\alpha}{\gamma(1-\alpha)}\right]^{\frac{\gamma(1-\alpha)}{\alpha(1-\gamma)}}\left[\frac{\bar{X}^{\alpha}(A_t^{\mathrm{U}})^{1-\alpha}}{(1-\gamma)k_t^{\alpha+\frac{\gamma}{1-\gamma}}}\right]^{\frac{1}{\alpha}}\left[\left(1+\frac{\gamma(1-\alpha)}{\alpha}\right)k_t^{\frac{1}{1-\gamma}} - \gamma\beta k_t^{\alpha+\frac{\gamma}{1-\gamma}}\right] \\
&\equiv \psi^{(2)}(A_t^{\mathrm{U}}, k_t) \qquad (62)
\end{aligned}
$$

(62) shows that $\psi^{(2)}(A_t^{\mathrm{U}}, k_t) > 0$ implies $\tau^{(2)}(k_t) > 0$, which implies the growth of k_t. Thus $\psi^{(2)}(A_t^{\mathrm{U}}, k_t) > 0$ is a sufficient condition for k_t to grow over time. Similar to the previous regime, we now examine the sign of $\psi^{(2)}(A_t^{\mathrm{U}}, k_t)$.

Suppose that $\psi^{(2)}(A_t^{\mathrm{U}}, k_t)$ is positive when the economy moves into this regime. That is, we have $\psi^{(2)}(A_t^{\mathrm{U}}, k_t) > 0$ when $k_t = \underline{k}$,[11] with $\underline{k}$ being defined in (35). Also we can verify that given A_t^{U}, $\lim_{k_t\to+\infty}\psi^{(2)}(A_t^{\mathrm{U}}, k_t) = -\infty$ holds. Thus there exists $k_{2,t}^{*} = k_2(A_t^{\mathrm{U}})$ and $\eta_{2,t} > 0$ such that

$$
\begin{aligned}
&\psi^{(2)}(A_t^{\mathrm{U}}, k_{2,t}^{*}) = 0 \\
&\psi^{(2)}(A_t^{\mathrm{U}}, k_t) > 0 \quad \underline{k} \le k_t < k_{2,t}^{*} \\
&\psi^{(2)}(A_t^{\mathrm{U}}, k_t) < 0 \quad k_t > k_{2,t}^{*}
\end{aligned}
$$

In this way we can define $k_{2,t}^{*}$ as:

$$k_{2,t}^{*} = \inf\{\underline{k} < k_t < \bar{k} | \psi^{(2)}(A_t^{\mathrm{U}}, k_t) \le 0\} \qquad (63)$$

With $k_{2,t}^{*}$ being defined in (63), we can show that k_t continues to grow as a result of technological progress in this regime in the following proposition.

Proposition 4. *The growth of A_t^{U} results in continuous growth of k_t in the early modern regime.*

Proof. According to equation (54) in assumption 5, we have

$$\left(1+\frac{\gamma(1-\alpha)}{\alpha}\right)k_t^{\frac{1}{1-\gamma}} - \gamma\beta k_t^{\alpha+\frac{\gamma}{1-\gamma}} > 0$$

[11] If not, we have $\psi^{(2)}(A_t^{\mathrm{U}}, \underline{e}) < 0$. Then $\tau^{(2)}(A_t^{\mathrm{U}}, k_t)$ may be either greater than or less than zero. In case that $\tau^{(2)}(A_t^{\mathrm{U}}, k_t) < 0$ is less than zero, then k_{t+1} will shrink to the previous regime. But this is only temporary. As A_t^{U} grows to a substantially high level, $\psi^{(2)}(A_t^{\mathrm{U}}, k_t)$ will eventually become positive at $k_t = \underline{k}$, bringing the economy back to this regime. So it is reasonable to directly suppose $\psi^{(2)}(A_t^{\mathrm{U}}, \underline{k}) > 0$.

holds when $k_t = \underline{k} = \alpha/\gamma(1-\alpha)$. As $\left(1+\frac{\gamma(1-\alpha)^2}{\alpha}\right)k_t - \gamma\beta(1-\alpha)k_t^{\alpha}$ is increasing in k_t, it is positive for any $k_t > \underline{k}$. In this way, we have:

$$\frac{\partial\psi^{(2)}(A_t^{\mathrm{U}}, k_t)}{\partial A_t^{\mathrm{U}}} = \left[\frac{\alpha}{\gamma(1-\alpha)}\right]^{\frac{\gamma(1-\alpha)}{\alpha(1-\gamma)}} \left[\frac{\bar{X}^{\alpha}}{(1-\gamma)k_t^{\alpha+\frac{\gamma}{1-\gamma}}}\right]^{\frac{1}{\alpha}} \left[\left(1+\frac{\gamma(1-\alpha)}{\alpha}\right)k_t^{\frac{1}{1-\gamma}} - \gamma\beta k_t^{\alpha+\frac{\gamma}{1-\gamma}}\right]$$
$$\frac{1-\alpha}{\alpha}(A_t^{\mathrm{U}})^{\frac{1-2\alpha}{\alpha}} > 0$$

As defined in equation (63), $k_{2,t}^*$ denotes the minimum k_t to maintain $\psi^{(2)}(A_t^{\mathrm{U}}, k_t) \leq 0$. Then according to theorem 1 in Milgrom and Roberts (1994), $\partial\psi^{(2)}(A_t^{\mathrm{U}}, k_t)/\partial A_t^{\mathrm{U}} > 0$ implies growing A_t^{U} causes $k_{2,t}^*$ to increase. So there exists $T_2 > T_1$ such that when $t \geq T_2$ holds, $k_{2,t}^*$ goes beyond $\bar{k}$, the threshold that distinguishes the early modern regime from the modern growth regime. Then $\psi^{(2)}(A_t^{\mathrm{U}}, k_t) > 0$ holds for all $\underline{k} < k_t < \bar{k} < k_{2,t}^*$.

According to (62), $\tau^{(2)}(k_t) > \psi^{(2)}(A_t^{\mathrm{U}}, k_t)$ holds. Then at the time when $t \geq T_2$ holds, $\tau^{(2)}(k_t) > \psi^{(2)}(A_t^{\mathrm{U}}, k_t) > 0$ holds for all $\underline{k} < k_t < \bar{k}$. The positive value of $\tau^{(2)}(k_t)$ results in the continual growth of k_t. □

Proposition 4 shows continual growth of the capital-human capital ratio over time. This indicates the co-existence of constant skill premium and continual growth of capital-human capital ratio. As can be seen from (60), the human capital investment and the human capital accumulated by each individual are increasing in the capital-human capital ratio. Proposition 4 thus implies that the human capital investment and accumulation are growing mildly in this regime. This shows that the economy in western Europe already grows beyond the "bare bone subsistence level", as empirically shown by Broadberry et. al (2015). This finding also consists with Foreman-Peck and Zhou (2016), who document positive growth in human capital accumulation in this period. Despite its mild growth, the human capital investment is still not powerful enough to spur the growth of the productivity of human capital, which distinguishes this regime from the modern period.

Eventually the capital-human capital ratio grows to a sufficiently high level (i.e.$k_{t+1} > \bar{k}$) and the economy takes off into the modern growth regime. How such takes off and the continuation growth in the modern regime affect the skill premium will be analyzed subsequently.

4.3. **Skill Premium in the Modern Growth Regime:1800-1914.** As the capital-human capital ratio becomes sufficiently high (i.e. $k_{t+1} \geq \bar{k}$), the economy takes off into the modern growth regime. The human capital investment made by individuals is large enough to augment the growth of the productivity of human capital (i.e. positive growth rate of A_{t+1}). As is shown by (32), we can write the capital-human capital ratio as an inverse function of the human capital investment. The human capital per skilled worker h_{t+1}

satisfies $h_{t+1} = e_t^{\gamma}$. In this way , we can write k_{t+1} and h_{t+1} in terms of e_t as:

$$k_{t+1} = e_t^{1-\gamma}\left(1 - \frac{1}{e_t^{\gamma}}\right)\frac{\alpha}{\gamma(1-\alpha)} \equiv k(e_t) \quad h_{t+1} = e_t^{\gamma} \tag{64}$$

As the human capital investment generates positive growth rate of A_{t+1}, A_{t+1} is no longer constant. Then the optimal supply of unskilled labour U_{t+1} is determined by A_{t+1}, A^{U}_{t+1} and k_{t+1}. According to (64), k_{t+1} is a function of e_t (i.e. $k_{t+1} = k(e_t)$). So it is equivalent to claim that U_{t+1} is determined by A_{t+1}, A^{U}_{t+1} and e_t. Then U_{t+1} can be denoted as $U_{t+1} = U^{(3)}(A_{t+1}, A^{\mathrm{U}}_{t+1}, e_t)$. Using equation (42), we can write $U(A_{t+1}, A^{\mathrm{U}}_{t+1}, e_t)$ as:

$$U^{(3)}(A_{t+1}, A^{\mathrm{U}}_{t+1}, e_t) = \left[\frac{(1-\alpha)\bar{X}^{\alpha}(A^{\mathrm{U}}_{t+1})^{1-\alpha}}{(1-\alpha)A_{t+1}^{1-\alpha}k(e_t)^{\alpha}e_t^{\gamma} - \alpha e_t A_{t+1}^{1-\alpha}k(e_t)^{\alpha-1}}\right]^{\frac{1}{\alpha}} \tag{65}$$

Because we can not write e_t as an explicit function of k_{t+1}, we instead analyse the dynamics of e_t to see the evolution of the skill premium in this regime. To do this, we first transform the $\tau(k_t)$ in (49) into $\tau(k_{t+1})$ as[12]:

$$\begin{aligned}\tau(k_{t+1}) &= L(\beta k_{t+1}^{\alpha}A_{t+1}^{1-\alpha}h_{t+1} - k_{t+1}h_{t+1} - e_t) + U(A_{t+2}, A^{\mathrm{U}}_{t+2}, k_{t+1})(k_{t+1}h_{t+1} + e_t) \\ &- U(A_{t+1}, A^{\mathrm{U}}_{t+1}, k_{t+1})\frac{\alpha e_t \beta k_{t+1}^{\alpha-1}A_{t+1}^{1-\alpha}}{1-\alpha} \equiv \tau^{(3)}(e_t)\end{aligned} \tag{66}$$

In (66), we have $U(A_{t+1}, A^{\mathrm{U}}_{t+1}, k_{t+1}) = U^{(3)}(A_{t+1}, A^{\mathrm{U}}_{t+1}, e_t)$. And the term $U(A_{t+2}, A^{\mathrm{U}}_{t+2}, k_{t+1})$ is derived by changing the terms A_{t+1} and A^{U}_{t+1} on the right hand side of (65) into A_{t+2} and A^{U}_{t+2}, respectively. $U(A_{t+2}, A^{\mathrm{U}}_{t+2}, k_{t+1})$ can be denoted as $U^{(3)}(A_{t+2}, A^{\mathrm{U}}_{t+2}, e_t)$. Plug the formulations of k_{t+1} and h_{t+1} in terms of e_t displayed by (64) into (66), we can rewrite $\tau^{(3)}(e_t)$ as:

$$\begin{aligned}\tau^{(3)}(e_t) &= L(\beta k(e_t)^{\alpha}A_{t+1}^{1-\alpha}e_t^{\gamma} - k(e_t)e_t^{\gamma} - e_t) + U^{(3)}(A_{t+2}, A^{\mathrm{U}}_{t+2}, e_t)(k(e_t)e_t^{\gamma} + e_t) \\ &- U^{(3)}(A_{t+1}, A^{\mathrm{U}}_{t+1}, e_t)\frac{\alpha e_t \beta k(e_t)^{\alpha-1}A_{t+1}^{1-\alpha}}{1-\alpha}\end{aligned} \tag{67}$$

According to the property of τ, $\tau(k_{t+1}) > 0$ results in $k_{t+2} > k_{t+1}$. Equivalently, given $\tau^{(3)}(e_t) > 0$, we have $k_{t+2} > k_{t+1}$. Moreover, the formulation of k_{t+1} in terms of e_t in (64) indicates that k_{t+1} is increasing in e_t, which means that $k_{t+2} > k_{t+1}$ implies $e_{t+1} > e_t$. In this way, $\tau^{(3)}(e_t) > 0$ implies that $e_{t+1} > e_t$. To check if e_t is growing is equivalent to checking if the sign of $\tau^{(3)}(e_t)$ is positive.

According to canonical long-run growth literature such as Galor and Weil (2000), the main feature of the economy in the modern growth regime is "sustainable growth", with constant technological progress powered by human capital investment. In our framework, this feature is reflected by constant growth of the productivity of human capital A_{t+1}. On the other hand, from the formulation of the growth rate of A_{t+1} in (21), we can see that when $e_t \to +\infty$, the growth rate of A_{t+1}, g_{t+1} converges to a constant level of $\bar{g} - 1$. The continuation of the growth of the human capital investment is thus the key to maintaining

[12]This is done by replacing the subscript t on the right hand side of the second equality of (49) with $t + 1$

the sustainable growth of the economy. Whether the human capital investment is growing depends on whether $\tau^{(3)}(e_t) > 0$ holds, with $\tau^{(3)}(e_t)$ being formulated in (67), we thus analyze the sign of $\tau^{(3)}(e_t)$.

Note that from equation (67) we can derive the following inequality:

$$\tau^{(3)}(e_t) > L(\beta k(e_t)^\alpha A_{t+1}^{1-\alpha} e_t^\gamma - k(e_t)e_t^\gamma - e_t) - U^{(3)}(A_{t+1}, A_{t+1}^{\mathrm{U}}, e_t)\frac{\alpha e_t \beta k(e_t)^{\alpha-1} A_{t+1}^{1-\alpha}}{1-\alpha}$$
$$= L(\beta k(e_t)^\alpha A_{t+1}^{1-\alpha} e_t^\gamma - k(e_t)e_t^\gamma - e_t) -$$
$$\left(\frac{A_{t+1}^{1-\alpha}}{A_{t+1}^{\mathrm{U}}}\right)^{\frac{\alpha-1}{\alpha}} \left[\frac{(1-\alpha)\bar{X}^\alpha}{(1-\alpha)k(e_t)^\alpha e_t^\gamma - \alpha e_t k(e_t)^{\alpha-1}}\right]^{\frac{1}{\alpha}} \frac{\alpha e_t \beta k(e_t)^{\alpha-1}}{1-\alpha} \equiv \psi^{(3)}\left(A_{t+1}, \frac{A_{t+1}^{1-\alpha}}{A_{t+1}^{\mathrm{U}}}, e_t\right) \quad (68)$$

Denote the term $\psi^{(3)}\left(A_{t+1}, \frac{A_{t+1}^{1-\alpha}}{A_{t+1}^{\mathrm{U}}}, e_t\right)$ in (68) as $\psi^{(3)}$ for short. Then if $\psi^{(3)}$ is greater than zero, this implies $\tau^{(3)}(e_t) > 0$, equivalently $e_{t+1} > e_t$. So $\psi^{(3)} > 0$ is the sufficient condition of maintaining growing human capital investment. We can then analyse the evolution of e_t by examining the sign of $\psi^{(3)}$.

As discussed before, when e_t becomes sufficiently high, equation (52) holds. In this way, we can see that for sufficiently large e_t, $A_{t+1}^{1-\alpha}$ grows faster than A_{t+1}^{U}. That is, the ratio $A_{t+1}^{1-\alpha}/A_{t+1}^{\mathrm{U}}$ is increasing over time. In this way, we can divide the possible outcomes regarding $\psi^{(3)}$ into two categories:

The first type of outcome is that there exists $T_3 > 0 (T_3 > T_2 > T_1)$ such that when $t = T_3$, we have $\psi^{(3)} > 0$ as well as growing $A_{t+1}^{1-\alpha}/A_{t+1}^{\mathrm{U}}$. This case will generate continuing growth of human capital investment, leading the growth rate of the productivity of human capital g_{t+1} to converge to a stable and high level $\bar{g} - 1$. Thus a sustainable growth path is generated. This case can then be referred to as a "good case".

At $t = T_3$, $e_t = e_{T_3}$ holds and we have $\psi^{(3)} = \psi^{(3)}\left(A_{t+1}, \frac{A_{t+1}^{1-\alpha}}{A_{t+1}^{\mathrm{U}}}, e_{T_3}\right) > 0$. Note that from (68) we can see that

$$\lim_{e_t \to +\infty} \psi^{(3)} = -\infty$$

holds for given A_{t+1} and A_{t+1}^{U}. So there exists $e_{3,t}^*$ such that

$$\begin{aligned}
&\psi^{(3)}\left(A_{t+1}, \frac{A_{t+1}^{1-\alpha}}{A_{t+1}^{\mathrm{U}}}, e_{3,t}^*\right) = 0 \\
&\psi^{(3)}\left(A_{t+1}, \frac{A_{t+1}^{1-\alpha}}{A_{t+1}^{\mathrm{U}}}, e_t\right) > 0 \qquad e_{T_3} \leq e_t < e_{3,t}^* \\
&\psi^{(3)}\left(A_{t+1}, \frac{A_{t+1}^{1-\alpha}}{A_{t+1}^{\mathrm{U}}}, e_t\right) < 0 \quad e_{3,t}^* < e_t < e_{3,t}^* + \eta \quad \eta > 0
\end{aligned}$$

In this way we can define $e_{3,t}^*$ as:

$$e_{3,t}^* = \inf\left\{e_t > e_{T_3} | \psi^{(3)}\left(A_{t+1}, \frac{A_{t+1}^{1-\alpha}}{A_{t+1}^{\mathrm{U}}}, e_t\right) \leq 0\right\} \quad (69)$$

With (69), we can show how the "good case" generates a sustainable growth path in the following proposition:

Proposition 5. *In the "good case" as described above, the human capital investment e_t will grow continually in the long run (i.e. $lim_{t\to+\infty}e_t = +\infty$). As a result, the growth rate of the productivity of human capital, g_{t+1}, will converge to a high and constant level(i.e. $lim_{t\to+\infty}g_{t+1} = \bar{g} - 1$)*

Proof. We can derive the following from equation (68)

$$\frac{\partial\psi^{(3)}}{\partial A_{t+1}} = L(1-\alpha)\beta k(e_t)^\alpha e_t^\gamma A_{t+1}^{-\alpha} > 0 \tag{70}$$

and

$$\begin{aligned}\frac{\partial\psi^{(3)}}{\partial\frac{A_{t+1}^{1-\alpha}}{A_{t+1}^{\mathrm{U}}}} &= (-1)\frac{\alpha-1}{\alpha}\left(\frac{A_{t+1}^{1-\alpha}}{A_{t+1}^{\mathrm{U}}}\right)^{\frac{-1}{\alpha}}\left[\frac{(1-\alpha)\bar{X}^\alpha}{(1-\alpha)k(e_t)^\alpha e_t^\gamma - \alpha e_t k(e_t)^{\alpha-1}}\right]^{\frac{1}{\alpha}}\frac{\alpha e_t\beta k(e_t)^{\alpha-1}}{1-\alpha}\\ &= \left(\frac{A_{t+1}^{1-\alpha}}{A_{t+1}^{\mathrm{U}}}\right)^{\frac{-1}{\alpha}}\left[\frac{(1-\alpha)\bar{X}^\alpha}{(1-\alpha)k(e_t)^\alpha e_t^\gamma - \alpha e_t k(e_t)^{\alpha-1}}\right]^{\frac{1}{\alpha}} e_t\beta k(e_t)^{\alpha-1} > 0 \end{aligned}\tag{71}$$

(70) and (71) show that $\psi^{(3)}$ is increasing in both A_{t+1} and $A_{t+1}^{1-\alpha}/A_{t+1}^{\mathrm{U}}$.

According to the characterization of the "good case", when $t = T_3$, we have

$$\psi^{(3)} = \psi^{(3)}\left(A_{t+1}, \frac{A_{t+1}^{1-\alpha}}{A_{t+1}^{\mathrm{U}}}, e_{T_3}\right) > 0 \tag{72}$$

Then according to Theorem 1 in Milgrom and Roberts (1994), (70), (71) and (72) imply that the $e_{3,t}^*$ which is formulated in (69) is increasing as A_{t+1} and $A_{t+1}^{1-\alpha}/A_{t+1}^{\mathrm{U}}$ are growing. With A_{t+1} and $A_{t+1}^{1-\alpha}/A_{t+1}^{\mathrm{U}}$ growing constantly, $e_{3,t}^*$ will grow all the way to infinity in the long run. This results in $\psi^{(3)} > 0$ for all $e_t > e_{T_3}$.

On the other hand, as $\tau^{(3)}(e_t) > \psi^{(3)}$, $\psi^{(3)} > 0\,\forall\, e_t > e_{T_3}$ implies that $\tau^{(3)}(e_t) > \psi^{(3)} > 0$ holds for all $e_t > e_{T_3}$. This will then result in continuous growth of e_t. As e_t goes to infinity, (21) indicates that

$$\lim_{e_t\to+\infty}g_{t+1} = \lim_{e_t\to+\infty}\bar{g}\left(1-\frac{1}{e_t^\gamma}\right) - 1 = \bar{g} - 1$$

The equation above shows that the growth rate of the productivity of human capital will converge to a constant level $\bar{g} - 1$, therefore a sustainable growth is maintained. □

As the human capital investment is deducted from the bequest of each individual, one may wonder if the human capital investment like this can be maintained. The answer is yes, as can be seen in the following corollary:

Corollary 1. *The bequest inherited by each individual is feasible to maintain as large human capital investment as possible.*

Proof. According to equation (18), the bequest the individual i receives in t, b_t^{i}, satisfies $b_t^{\mathrm{i}} = \beta I_t^{\mathrm{i}}$. Because the second period income I_t^{i} is identical across individuals, the bequest each individual receives is identical. On the other hand, as discussed in 2.3.2, aggregate bequest in period t, $\sum_i \beta I_t^{\mathrm{i}} = \beta Y_t$. Therefore given population L, an individual's bequest satisfies:

$$b_t^{\mathrm{i}} = \frac{\beta Y_t}{L}$$

Using equation (44), we can derive b_t^{i} as:

$$\begin{aligned} b_t^{\mathrm{i}} &= \frac{\beta Y_t}{L} = \frac{K_{t+1} + e_t S_{t+1}}{L} = \frac{S_{t+1} k_{t+1} h_{t+1} + S_{t+1} e_t}{L} \\ &= \frac{S_{t+1}}{L}(k_{t+1} h_{t+1} + e_t) = \frac{L - U_{t+1}}{L}(k(e_t) e_t^{\gamma} + e_t) \end{aligned}$$

So the ratio b_t^{i}/e_t satisfies:

$$\frac{b_t^{\mathrm{i}}}{e_t} = \frac{L - U_{t+1}}{L}(k(e_t) e_t^{\gamma-1} + 1) = \frac{L - U_{t+1}}{L}\left[\left(1 - \frac{1}{e_t^{\gamma}}\right)\frac{\alpha}{\gamma(1-\alpha)} + 1\right] \tag{73}$$

In (73), $U_{t+1} = U^{(3)}(A_{t+1}, A_{t+1}^{\mathrm{U}}, e_t)$. We can then write U_{t+1} as:

$$U_{t+1} = U^{(3)}(A_{t+1}, A_{t+1}^{\mathrm{U}}, e_t) = \left[\frac{(A_{t+1}^{\mathrm{U}})^{1-\alpha}}{(A_{t+1})^{1-\alpha} k(e_t)^{\alpha} e_t^{\gamma}}\right]^{\frac{1}{\alpha}} \left[\frac{(1-\alpha)\bar{X}^{\alpha}}{(1-\alpha) - \alpha e_t^{1-\gamma} k(e_t)^{-1}}\right]^{\frac{1}{\alpha}}$$

Using the formulation of $k(e_t)$ in (64), we have:

$$e_t^{1-\gamma} k(e_t)^{-1} = e_t^{1-\gamma} e_t^{\gamma-1}\left(1 - \frac{1}{e_t^{\gamma}}\right)^{-1} \frac{\gamma(1-\alpha)}{\alpha} = \frac{e_t^{\gamma}}{e_t^{\gamma} - 1}\frac{\gamma(1-\alpha)}{\alpha}$$

So we have $\lim_{e_t \to +\infty} e_t^{1-\gamma} k(e_t)^{-1} = \gamma(1-\alpha)/\alpha$.

On the other hand, from the formulation of $k(e_t)$ in (64) we have $\lim_{e_t \to +\infty} k(e_t) = +\infty$. And as implied by assumption (4), for sufficiently large e_t, A_{t+1} grows faster than A_{t+1}^{U}. So we have $\lim_{e_t \to +\infty} A_{t+1}^{\mathrm{U}}/A_{t+1} = 0$. In this way, we have

$$\begin{aligned} &\lim_{e_t \to +\infty} U_{t+1} = \lim_{e_t \to +\infty}\left[\frac{(A_{t+1}^{\mathrm{U}})^{1-\alpha}}{(A_{t+1})^{1-\alpha} k(e_t)^{\alpha} e_t^{\gamma}}\right]^{\frac{1}{\alpha}} \lim_{e_t \to +\infty}\left[\frac{(1-\alpha)\bar{X}^{\alpha}}{(1-\alpha) - \alpha e_t^{1-\gamma} k(e_t)^{-1}}\right]^{\frac{1}{\alpha}} \\ &= 0 * \left(\frac{\bar{X}^{\alpha}}{1-\gamma}\right)^{\frac{1}{\alpha}} = 0 \end{aligned}$$

In this way, when $e_t \to +\infty$, the ratio of bequest and human capital investment satisfies:

$$\lim_{e_t \to +\infty} \frac{b_t^{\mathrm{i}}}{e_t} = \lim_{e_t \to +\infty} \frac{L - U_{t+1}}{L} \lim_{e_t \to +\infty}\left[\left(1 - \frac{1}{e_t^{\gamma}}\right)\frac{\alpha}{\gamma(1-\alpha)} + 1\right] = 1 + \frac{\alpha}{\gamma(1-\alpha)} > 1 \tag{74}$$

(74) shows that the bequest each individual receives grows at the same pace as the human capital investment. It also shows that as the human capital investment goes to infinity, the bequest tends to stay at a level higher than human capital investment. This allows for unbounded growth of human capital investment.

□

In another case, $\psi^{(3)} < 0$ always holds. It can then be verified that a sustainable growth of e_t cannot be maintained. This is because if there is a sustainable growth of e_t, then as

e_t becomes fairly large, the term $U^{(3)}(A_{t+2}, A^{\text{U}}_{t+2}, e_t)(k(e_t)e_t^{\gamma} + e_t)$ in (67) will become fairly small, resulting in the sign of $\tau(k_t)^{(3)}$ identical to that of $\psi^{(3)}$. This will result in negative sign of $\tau(k_t)^{(3)}$, bringing down the human capital investment e_t. As e_t fails to grow continually, the growth rate of the human capital productivity g_{t+1} does not converge to the high level of $\bar{g} - 1$. This case can thus be referred to as the "bad case".

Obviously the economy in western Europe is in the "good case" as it is constantly growing at a higher rate than other Eurasian regions. We will then focus on the skill premium in the "good" case. The following proposition applies:

Proposition 6. *Given that the economy is on the sustainable growth path, the skill premium will first go up , then converge back to the same level as in early modern regime.*

Proof. Plugging (64) into the formulation of the skill premium in equation (55), we can write the skill premium in terms of e_t as:

$$\left(\frac{w^{\text{S}}_{t+1}}{w^{\text{U}}_{t+1}}\right)^{(3)} = \frac{1}{1 - \frac{\alpha}{1-\alpha}\frac{e_t}{k_{t+1}h_{t+1}}} = \frac{1}{1 - \frac{\alpha}{1-\alpha}\frac{e_t^{1-\gamma}}{k(e_t)}} = \frac{1}{1 - \frac{\gamma e_t^{\gamma}}{e_t^{\gamma}-1}} \tag{75}$$

From (75), the following result immediately shows up:

$$\left(\frac{w^{\text{S}}_{t+1}}{w^{\text{U}}_{t+1}}\right)^{(3)} = \frac{1}{1 - \frac{\gamma e_t^{\gamma}}{e_t^{\gamma}-1}} > \frac{1}{1-\gamma} = \left(\frac{w^{\text{S}}_{t+1}}{w^{\text{U}}_{t+1}}\right)^{(2)} \tag{76}$$

(76) shows that in the modern growth regime, the skill premium is higher than the previous regime in the short run.

In the long run, however, the "good case" indicates that $\lim_{t\to+\infty} e_t = +\infty$ holds. Then we can derive the long-run skill premium as:

$$\lim_{t\to+\infty}\left(\frac{w^{\text{S}}_{t+1}}{w^{\text{U}}_{t+1}}\right)^{(3)} = \lim_{e_t\to+\infty}\left(\frac{w^{\text{S}}_{t+1}}{w^{\text{U}}_{t+1}}\right)^{(3)} = \lim_{e_t\to+\infty}\frac{1}{1 - \frac{\gamma e_t^{\gamma}}{e_t^{\gamma}-1}}$$

$$= \frac{1}{1-\gamma} = \left(\frac{w^{\text{S}}_{t+1}}{w^{\text{U}}_{t+1}}\right)^{(2)} \tag{77}$$

(77) shows that in the long run, as human capital investment continues to grow, the skill premium converges back to the original level in the early modern regime.

□

According to proposition 6, the skill premium in the modern growth regime first goes up. Then with the continual growth of the human capital investment, the skill premium starts to decline and converge back to the same level as in the previous regime. Thus the overall level of the skill premium in this regime does not vary much from the previous regime.

5. Conclusion and Discussion

This paper theoretically analyses the evolution of the skill premium in western Europe from 1300 to 1914, a period that stretches from pre-modern society to modern society. To do this, we build a long-run growth model that endogenously generates different growth regimes in a way similar to the unified growth model in the canonical long-run growth literature. We show that the growth of the capital-human capital ratio and the human capital investment play a key role in shaping the skill premium. The growth of the human capital investment has a positive impact on the skill premium while that of the capital-human capital ratio has a negative impact. Which one dominates the other depends on the level of the capital-human capital ratio.

The process of development is featured with growing capital-human capital ratio over time. When the capital-human capital ratio is low, the negative effect of growing capital-human capital ratio dominates. As the capital-human capital ratio grows to a higher level, the positive effect of the human capital investment cancels out with the negative effect of growing capital-human capital ratio. But when the capital-human capital ratio goes beyond a sufficiently high level, the negative effect of growing capital-human capital ratio becomes dominant again. In this way, as the capital-human capital ratio grows from an initially low level to a sufficiently high level, the process of development is partitioned into three different regimes: the late medieval regime (circa 1300 to 1600), the early modern regime (circa 1600 to 1800) and the modern growth regime (circa 1800 to 1914). As the economic growth and transition take place in and across these regimes, the skill premium will exhibit the "first declining then stable" pattern as is shown in Figure I.

This paper successfully explains the evolution of the skill premium in western Europe in the very long run. Our findings contribute to both the literature on the skill premium and that on the long-run growth. We show that the economic development and the technological change in history, after incurring an initial fall of the skill premium, lead the skill premium to converge to a low and stable level. This is of contrast to the contemporary SBTC, which pushes the skill premium upward. On the other hand, our findings reveals how the economic growth and the technological change in the very long run affect inequality. In particular, we find that the growth of the human capital investment and the capital-human capital ratio not only drives the economic growth in the very long run, but affects the skill premium as well. This finding of the influence that long-run growth and development have on the wage inequality is new to the long-run growth literature.

The generation of the three regimes of growth and endogenous transition from one regime to another are crucial to analysing the evolution of the skill premium in the very long run. In the future, we could possibly bring the model into data and see whether these three different growth regimes can be generated and whether there is endogenous transition across them. We would then examine if the simulated trajectory of the skill premium across different regimes of growth is consistent with the actual evolution of the skill premium. By bringing the model into data, we will test whether the prediction of our model is consistent with the real world.

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