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EGALITARIAN EQUIVALENCE AND STRATEGYPROOFNESS IN THE QUEUEING PROBLEM

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Working Paper No. 13/16

July 2013

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ABSTRACT. We investigate the implications of egalitarian equivalence (Pazner and Schmeidler [22]) together with queue efficiency and strategyproofness in the context of queueing problems. We completely characterize the class of mechanisms satisfying the three requirements. Though there is no mechanism in this class satisfying budget balance, feasible mechanisms exist. We also show that it is impossible to find a mechanism satisfying queue efficiency, egalitarian equivalence and a stronger notion of strategyproofness called weak group strategyproofness. In addition, we show that generically there is no mechanism satisfying two normative notions, egalitarian equivalence and no-envy, together.

JEL Classifications: C72, D63, D82.

Keywords: Queueing problem, queue efficiency, strategyproofness, egalitarian equivalence, budget balance, feasibility, weak group strategyproofness, no-envy.

1. INTRODUCTION

There is a significant recent literature in economics on queueing models, both from a normative viewpoint (Chun [4], [5], Maniquet [15]) as well as from a strategic viewpoint (Kayi and Ramaekers [14], Mitra [16], Mitra and Mutuswami [17]).[1] Our objective in this paper is to combine the strategic and normative approaches; in particular, we are interested in mechanisms that have nice strategic and normative

Date: April 8, 2013.

This paper is based on the second half of our paper entitled "No-envy, egalitarian equivalence and strategyproofness in queueing problems." The authors are grateful to William Thomson for his comments and suggestions. Chun's work was supported by the National Research Foundation of Korea Grant funded by the Korean Government KRF-2009-342-B00011 and NRF-2010-330-B00077.

[1]There is an extensive literature on queueing in Operations Research. Furthermore, at least some of the recent work explicitly uses Game Theory. That literature is too vast to survey here and we refer the interested reader to Hassin and Haviv [12].

properties for the queueing problem with a single server, linear waiting costs and quasi-linear preferences.[2]

The strategic notion we use is strategyproofness which requires that an agent should not have an incentive to misrepresent her waiting cost no matter what she believes other agents to be doing. For the normative notion, we have at least two well-known alternatives: *no-envy* and *egalitarian equivalence*. No-envy (Foley [10]) requires that no agent should prefer consuming another agent's allocation. It has been analyzed in many different contexts (Alkan, Demange and Gale [1], Pàpai [21], Svensson [23], Tadenuma and Thomson [24], and Thomson [27]). In queueing problems, its implications have been studied by Chun [4] and Kayi and Ramaekers [14].

In this paper, we focus on the second normative notion, egalitarian equivalence (Pazner and Schmeidler [22]). An allocation is egalitarian equivalent if there is a reference bundle such that each agent is indifferent between her allocation and the reference bundle. Like no-envy, an attractive feature of egalitarian equivalence is that it is an ordinal concept and makes no inter-personal utility comparisons. Since the reference bundle is common, it is easy to see that it satisfies equal treatment of equals or horizontal equity. It has also been studied in many contexts (Demange [8], Dutta and Vohra [9], Thomson [25] and Yengin [29], [30]). However, to the best of our knowledge, egalitarian equivalence in the queueing context has not been studied so far.

We investigate the implications of imposing egalitarian equivalence together with queue efficiency and strategyproofness.[3] Our starting point is the classic result of Holmström [13] which implies that in our context, a mechanism satisfies queue efficiency and strategyproofness if and only if it is a Vickrey-Clarke-Groves (VCG) mechanism.[4] It follows that imposing an additional criteria like egalitarian equivalence gives us a subset of VCG mechanisms.

We present a complete characterization of the family of mechanisms satisfying queue efficiency, strategyproofness and egalitarian equivalence. To understand our

[2]Such a combined analysis of strategic and fairness properties has also been done by Atlamaz and Yengin [2], Kayi and Ramaekers [14], Pápai [21], and Yengin [29], [30].
[3]The same issue was addressed in the context of allocation of heterogeneous objects by Yengin [29], [30].
[4]The family of VCG rules are due to Vickrey [28], Clarke [7] and Groves [11].

characterization, note that a reference bundle is a queue position along with a corresponding transfer and can change across profiles. We show that if we impose queue efficiency and strategyproofness along with egalitarian equivalence, then there is effectively only one degree of freedom in choosing the position in the reference bundle. In particular, once a queue position is selected for a profile, then we must select the same position for all profiles. Furthermore, the transfers are determined up to a type independent constant which can be any real number.

We go on to show that none of these mechanisms are budget balanced. However, we do get a possibility result with the weaker notion of feasibility which allows a mechanism to run budget surpluses but not deficits. We characterize the nonempty set of mechanisms that satisfy queue efficiency, strategyproofness, egalitarian equivalence and feasibility. This class restricts the reference bundle to choose only the first queue position. However, there cannot be an upper bound that can be placed on the budget surplus that may result.

Another desirable property of a mechanism is immunity to manipulations by groups of agents. Call a mechanism *weak group strategyproof* if it is not possible for a group of agents to manipulate their reports in a manner which makes all of them strictly better-off. We show that if there are three or more agents, then we cannot find mechanisms satisfying queue efficiency, weak group strategyproofness and egalitarian equivalence.[5]

The contrast between the results obtained here and the results obtained in the literature using no-envy is striking. With regard to budget balance, Kayi and Ramaekers [14] show that there are mechanisms satisfying no-envy, strategyproofness and budget balance.[6] With regard to weak group strategyproofness, Chun, Mitra and Mutuswami [6] show that there are mechanisms satisfying no-envy and weak group strategyproofness.

The rather sharp contrast leads us to suspect that no-envy and egalitarian equivalence are incompatible and we show that this indeed is the case if there are at least four agents. In the queueing problem, this incompatibility was shown by Chun [4] in the class of all budget balance mechanisms. Incompatibility was also obtained by

[5]As a matter of fact, these mechanisms don't even satisfy pairwise strategyproofness which only requires immunity against deviations by coalitions of size at most two.

[6]As shown in Remark 5.5, no-envy implies queue efficiency.

Thomson [25] in allocation problems with indivisible goods and by Thomson [26] in the context of time division. The only compatibility result is due to Yengin [30] in the context of heterogenous goods assignment problem with the option that each agent may be assigned more than one object.

In what follows, the model is given in Section 2. The characterization result on queue efficienct, strategyproof and egalitarian equivalent mechanisms is given in Section 3. We discuss the consequence of additionally imposing budget balance in Section 4. In Section 5, we show that strategyproofness can not be strengthened to weak group strategyproofness, and also the relationship between no-envy and egalitarian equivalence. We conclude in Section 6.

2. The model

Let $N = \{1, \ldots, n\}$, $n \geq 2$, be the set of agents. Each agent has one job to process and the machine can process only one job at a time. Each job takes the same processing time and without loss of generality, this time is normalized to one. A *queue* is an onto function $\sigma : N \to \{1, \ldots, n\}$. Agent i's position in the queue is denoted σ_i. The *predecessors of i in the queue σ*, denoted $P_i(\sigma)$, is the set $\{j|\sigma_j < \sigma_i\}$. Similarly the *followers of i in the queue σ*, denoted $F_i(\sigma)$, is the set $\{j|\sigma_j > \sigma_i\}$. When the context is clear, we shall abuse notation slightly by dropping the dependence on σ and simply referring to P_i and F_i. The set of all possible queues is $\Sigma(N)$.

Each agent is identified with her *waiting cost per unit of time* $\theta_i > 0$. If agent i's queue position is σ_i, then she incurs a waiting cost of $(\sigma_i - 1)\theta_i$.[7] An agent's net utility depends on her waiting costs and the transfers she receives. We assume that preferences are quasi-linear and so,

$$u_i(\sigma_i, t_i, \theta_i) = -(\sigma_i - 1)\theta_i + t_i, \qquad t_i \in \Re.$$

Let $\theta = (\theta_i)_{i \in N} \in \Re^n_{++}$ be the *profile* of waiting costs of all agents.[8] For all profiles θ and all $i \in N$, let $\theta_{N\setminus\{i\}}$ be the profile of waiting costs of all agents except i. A *queueing problem* G is the tuple (N, θ).

[7]This assumes that no waiting cost is incurred while the job is being processed. This is without loss of generality.

[8]Here $\Re_{++}$ denotes the positive orthant of the real line.

A queue σ is *efficient for the profile* θ if

$$\sigma \in \text{argmin}_{\sigma' \in \Sigma(N)} \sum_{i \in N} (\sigma'_i - 1)\theta_i.$$

In words, a queue is efficient if it minimizes the aggregate waiting costs of the agents. It is straightforward to check that queue efficiency implies the following condition:

$$\theta_i > \theta_j \implies \sigma_i < \sigma_j.$$

One can easily establish the converse too. That is, if σ is any queue obeying the above condition, then it is efficient.

Let $E(\theta)$ be the set of efficient queues at the profile θ. This set is a singleton if the efficient queue is unique which will be the case if $\theta_i \neq \theta_j$ for all $i, j \in N, i \neq j$.

A *mechanism* $\mu = (\sigma, t)$ associates to each queueing problem G, a tuple $\mu \equiv (\sigma, t) \in \Sigma(N) \times \Re^n$ where σ is the queue and $t = (t_1, \ldots, t_n)$ is the vector of transfers to the agents. In much of what we do, we shall be holding the set of agents constant and changing the preference profile. We shall note the dependence on the preference profile θ by denoting the allocation as $\mu(\theta) = (\sigma(\theta), t(\theta))$. Agent i's own allocation will be denoted $\mu_i(\theta) = (\sigma_i(\theta), t_i(\theta))$.

Next we introduce two properties of mechanisms that are of interest. The first is *queue efficiency*, which requires that a mechanism should choose an efficient queue at every profile.

Definition 2.1. A mechanism $\mu = (\sigma, t)$ satisfies *queue efficiency* (EFF) if for all profiles θ, $\sigma(\theta) \in E(\theta)$.

Remark 2.2. Our definition of a mechanism associates a unique queue to every queueing problem. Since $E(\theta)$ can contain more than one element, queue efficiency implicitly assumes the existence of a *tie-breaking rule* which selects an efficient queue whenever there is more than one such queue. For our purposes, any tie-breaking rule will suffice.

Our strategic notion is *strategyproofness* which requires that an agent should not strictly gain by misrepresenting her waiting cost no matter what she believes other agents to be doing. Let $u_i(\sigma_i(\theta'), t_i(\theta'); \theta_i) = -(\sigma_i(\theta') - 1)\theta_i + t_i(\theta')$ be the utility of agent i when the announced profile is θ' and her own waiting cost is θ_i. Call two profiles θ and θ' *i-variants* if $\theta_j = \theta'_j$ for all $j \neq i$.

Definition 2.3. A mechanism $\mu = (\sigma, t)$ satisfies *strategyproofness* (SP) if for all $i \in N$, all i-variants θ, θ', $u_i(\sigma_i(\theta), t_i(\theta); \theta_i) \geq u_i(\sigma_i(\theta'), t_i(\theta'); \theta_i)$.

3. Egalitarian equivalence and strategyproofness

Our main normative notion is egalitarian equivalence which was introduced by Pazner and Schmeidler [22].

Definition 3.1. A mechanism $\mu = (\sigma, t)$ is *egalitarian equivalent* (EE) if for all profiles θ, there exists a reference bundle $(\sigma_0(\theta), t_0(\theta))$ such that for all $i \in N$, $u_i(\sigma_i(\theta), t_i(\theta); \theta_i) = u_i(\sigma_0(\theta), t_0(\theta); \theta_i)$.

Egalitarian equivalence is based on the idea that an allocation where everyone consumes the same "reference bundle" is trivially egalitarian, and so is any redistribution which makes everyone indifferent between her own allocation and the reference bundle. In general, we would label an allocation egalitarian equivalent if we can find *one* reference bundle such that everyone is indifferent between her allocation and the reference bundle.

Our characterization result builds on the classic work of Holmström [13] on VCG mechanisms.

Definition 3.2. A mechanism $\mu = (\sigma, t)$ is a VCG mechanism if the transfers are given by

$$\forall \theta, \forall i \in N : t_i(\theta) = -\sum_{j \in F_i(\sigma)} \theta_j + g_i(\theta_{N \setminus \{i\}}). \tag{3.1}$$

Holmström [13] showed that the VCG mechanisms are the only ones satisfying EFF and SP when preferences are quasi-linear and the domain of types is convex. Since the domain of types is $\Re^n_{++}$ in our context, it follows that the only mechanisms satisfying EFF and SP are the VCG mechanisms.

The following theorem characterizes the set of mechanisms satisfying EFF, SP and EE. It shows that in effect, we have effectively one degree of freedom in choosing the position in the reference position. In particular, once we select a queue position σ_0 for a profile, then we must select the same σ_0 for all profiles. Furthermore, the transfers are specified up to a profile-independent constant c.

Theorem 3.3. A mechanism $\mu = (\sigma, t)$ satisfies EFF, SP and EE if and only if there exist $\sigma_0 \in \{1, \ldots, n\}$ and $c \in \Re$ such that

$$\forall \theta \in \Re^n_{++}, \forall i \in N : t_i(\theta) = \sum_{j \in N \setminus \{i\}} (\sigma_0 - \sigma_j(\theta))\theta_j + c. \tag{3.2}$$

To prove this theorem we use the following lemma. Given a queue $\sigma(\theta)$ for the profile θ and $i \in N$, we define the queue $\sigma(\theta_{N\setminus\{i\}})$ as follows:

$$\sigma_j(\theta_{N\setminus\{i\}}) = \begin{cases} \sigma_j(\theta) - 1 & \text{if } \sigma_j(\theta) > \sigma_i(\theta), \\ \sigma_j(\theta) & \text{otherwise.} \end{cases}$$

In words, $\sigma(\theta_{N\setminus\{i\}})$ is the queue formed by removing agent i and moving all agents behind her up by one position. It is easy to see that $\sigma(\theta_{N\setminus\{i\}})$ is efficient in the $N \setminus \{i\}$ economy for the profile $\theta_{N\setminus\{i\}}$ if $\sigma(\theta)$ is efficient for the profile θ.

Lemma 3.4. A mechanism $\mu = (\sigma, t)$ satisfies EFF, SP and EE only if it is a VCG mechanism such that

$$\forall i \in N, \forall \theta_{N\setminus\{i\}} : g_i(\theta_{N\setminus\{i\}}) = \sum_{j \neq i} (\sigma_0 - \sigma_j(\theta_{N\setminus\{i\}}))\theta_j + c, \tag{3.3}$$

where $\sigma_0 \in \{1, \ldots, n\}$ and $c \in \Re$.

Remark 3.5. We can see the restriction imposed by EE if we compare (3.3) to the VCG transfers (3.1). In (3.1), g_i can be any arbitrary function of $\theta_{N\setminus\{i\}}$. In (3.3), it is *affine* with the coefficients of θ_j, $j \neq i$, being pinned down precisely by the choice of σ_0. The set of mechanisms which are EFF, SP and EE is thus a small subset of the set of VCG mechanisms.

Proof. Let θ be a profile and $(\sigma_0(\theta), t_0(\theta))$ the corresponding reference bundle. Suppose that the efficient queue (for the profile θ) is such that $\sigma_i(\theta) = i$, $i \in N$. This implies that the profile θ must be such that $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$. For EFF, SP and EE to hold together, the following condition must hold:

$$\forall i \in N : -(\sigma_i(\theta) - 1)\theta_i - \sum_{j \in F_i(\sigma)} \theta_j + g_i(\theta_{N\setminus\{i\}}) = -(\sigma_0(\theta) - 1)\theta_i + t_0(\theta).$$

The left-hand side of the above expression is the utility from a VCG mechanism and the right hand side is the utility from the egalitarian equivalence requirement.

We can write the above as

$$t_0(\theta) = (\sigma_0(\theta) - \sigma_i(\theta))\theta_i - \sum_{j \in F_i(\sigma)} \theta_j + g_i(\theta_{N \setminus \{i\}}). \tag{3.4}$$

Choose two agents $i, i+1$. Noting that $\sigma_i(\theta) = i$ for all i, we have, using (3.4),

$$(\sigma_0(\theta) - i)\theta_i - \sum_{j>i} \theta_j + g_i(\theta_{N \setminus \{i\}}) = (\sigma_0(\theta) - i - 1)\theta_{i+1} - \sum_{j>i+1} \theta_j + g_{i+1}(\theta_{N \setminus \{i+1\}}). \tag{3.5}$$

Hence,

$$(\sigma_0(\theta) - i)\theta_i + g_i(\theta_{N \setminus \{i\}}) = (\sigma_0(\theta) - i)\theta_{i+1} + g_{i+1}(\theta_{N \setminus \{i+1\}}) \tag{3.6}$$

Since g_i does not depend on θ_i and g_{i+1} does not depend on θ_{i+1}, it follows that

$$g_i(\theta_{N \setminus \{i\}}) = (\sigma_0(\theta) - i)\theta_{i+1} + f_{i,i+1}(\theta_{N \setminus \{i,i+1\}}), \tag{3.7}$$

$$g_{i+1}(\theta_{N \setminus \{i+1\}}) = (\sigma_0(\theta) - i)\theta_i + f_{i,i+1}(\theta_{N \setminus \{i,i+1\}}). \tag{3.8}$$

Note that (3.6) implies that the same function $f_{i,i+1}(\theta_{N \setminus \{i,i+1\}})$ appears in (3.7) and (3.8). By putting $i+1$ instead of i in (3.6), we obtain

$$g_{i+1}(\theta_{N \setminus \{i+1\}}) = (\sigma_0(\theta) - i - 1)\theta_{i+2} + f_{i+1,i+2}(\theta_{N \setminus \{i+1,i+2\}}). \tag{3.9}$$

Equating (3.8) and (3.9) gives us a recursive relationship between $f_{i,i+1}$ and $f_{i+1,i+2}$:

$$f_{i+1,i+2}(\theta_{N \setminus \{i+1,i+2\}}) = (\sigma_0(\theta) - i)\theta_i - (\sigma_0(\theta) - i - 1)\theta_{i+2} + f_{i,i+1}(\theta_{N \setminus \{i,i+1\}}). \tag{3.10}$$

Note that there are $n-1$ functions of the type $f_{i,i+1}$ and $n-2$ recursion relations. Hence, we can solve for $n-2$ of the $f_{i,i+1}$ functions in terms of one of them. Solving in terms of $f_{1,2}$ gives us:

$$\forall i = 2, \ldots, n-1 : f_{i,i+1}(\theta_{N \setminus \{i,i+1\}}) = f_{1,2}(\theta_{N \setminus \{1,2\}}) + (\sigma_0(\theta) - 1)\theta_2 - (\sigma_0(\theta) - i)\theta_{i+1}.$$

We are now left with the task of determining $f_{1,2}(\theta_{N \setminus \{1,2\}})$. Note that $f_{i,i+1}$ cannot depend on θ_i or θ_{i+1} while $-(\sigma_0(\theta) - i)\theta_{i+1}$ appears on the right-hand side. Noting that $f_{1,2}$ cannot depend on θ_1 or θ_2, it follows that

$$f_{1,2}(\theta_{N \setminus \{1,2\}}) = \sum_{i \geq 2} (\sigma_0(\theta) - i)\theta_{i+1} + c. \tag{3.11}$$

Observe that the constant is arbitrary. Furthermore, by the recursion relation (3.10), the same constant must appear in all $f_{i,i+1}$ functions and hence, in all g_i functions.

By substituting for $f_{1,2}$ in (3.7) (with $i = 1$), it follows that

$$g_1(\theta_{N\setminus\{1\}}) = (\sigma_0\theta) - 1)\theta_2 + \sum_{i\geq 2}(\sigma_0(\theta) - i)\theta_{i+1} + c = \sum_{i=1}^{n-1}(\sigma_0(\theta) - i)\theta_{i+1} + c. \tag{3.12}$$

Since $\sigma_{i+1}(\theta_{N\setminus\{1\}}) = i$ for $i = 1, \ldots, n-1$, the above can be written as

$$g_1(\theta_{N\setminus\{1\}}) = \sum_{j\neq 1}(\sigma_0(\theta) - \sigma_j(\theta_{N\setminus\{1\}}))\theta_j + c. \tag{3.13}$$

Since g_1 does not depend on θ_1, it follows that (3.13) also holds at any profile θ' such that $\theta'_{N\setminus\{1\}} = \theta_{N\setminus\{1\}}$. So, suppose that $\theta'_{N\setminus\{1\}} \neq \theta_{N\setminus\{1\}}$. Observe that we can always construct a profile θ' such that $\theta'_1 > \max_{i\neq 1}\theta'_i$. We can now use the same procedure to get (3.13). Since the procedure is symmetric, it follows that

$$\forall\theta, \forall i \in N : g_i(\theta_{N\setminus\{i\}}) = \sum_{j\neq i}(\sigma_0(\theta) - \sigma_j(\theta_{N\setminus\{i\}})\theta_j + c. \tag{3.14}$$

To complete the proof we show that for two different profiles θ and θ', the queue position in the reference bundle remains unchanged. For all $k \in \{0, 1, \ldots, n\}$, we define the profile θ^k by

$$\theta_i^k = \begin{cases} \theta'_k & \text{if } i \leq k, \\ \theta_k & \text{otherwise.} \end{cases}$$

Note that we are moving from $\theta = \theta^0$ to $\theta' = \theta^n$ by changing the waiting cost of agents, one at a time. It follows from (3.1) that $g_i(\theta^{i-1}_{N\setminus\{i\}}) = g_i(\theta^i_{N\setminus\{i\}})$. Since $\sigma_j(\theta^{i-1}_{N\setminus\{i\}}) = \sigma_j(\theta^i_{N\setminus\{i\}})$ for all $j \neq i$, it follows from (3.14) that

$$0 = g_i(\theta^{i-1}_{N\setminus\{i\}}) - g_i(\theta^i_{N\setminus\{i\}}) = \left[\sigma_0(\theta^{i-1}) - \sigma_0(\theta^i)\right] \sum_{j\in N\setminus\{i\}} \theta_j. \tag{3.15}$$

Since $\theta_j \in \Re_{++}$ for all $j \in N$, (3.15) implies that $\sigma_0(\theta^{i-1}) = \sigma_0(\theta^i)$. Hence $\sigma_0(\theta) = \sigma_0(\theta^0) = \sigma_0(\theta^1) = \cdots = \sigma_0(\theta^n) = \sigma_0(\theta')$ and the proof is complete. □

Proof of Theorem 3.3: Lemma 3.4 has an immediate useful implication which we will use. By substituting (3.3) in (3.4) and simplifying, it follows that

$$t_0(\theta) = \sum_{i\in N}(\sigma_0 - \sigma_i(\theta))\theta_i + c. \tag{3.16}$$

We first show the necessity of (3.2). By EE, $-(\sigma_i(\theta)-1)\theta_i+t_i(\theta) = -(\sigma_0(\theta)-1)\theta_i+t_0(\theta)$. Using (3.16) and Lemma 3.4, it follows that

$$t_i(\theta) = \sum_{j\neq i}(\sigma_0 - \sigma_j(\theta))\theta_j + c. \tag{3.17}$$

This establishes the necessity of (3.2). That σ_0 is any arbitrary queue position and c is an arbitrary constant follows from Lemma 3.4.

For sufficiency, consider μ such that the transfer satisfies (3.2), σ_0 is an arbitrary queue position and c an arbitrary real constant. EFF and SP are satisfied since μ is a VCG mechanism. We only need to check that it satisfies EE. Consider any $i \in N$. Using (3.17) we get $u_i(\mu_i(\theta);\theta_i) = -(\sigma_i(\theta)-1)\theta_i - (\sigma_0 - \sigma_i(\theta))\theta_i + t_0(\theta) = -(\sigma_0 - 1)\theta_i + t_0(\theta) = u_i(\sigma_0, t_0(\theta);\theta_i)$. Since the selection of i was arbitrary, EE follows. □

4. Budget balance and feasibility

We now examine whether the mechanisms that we have identified satisfy additional desirable properties. One such property is *budget balance*. This requires that the sum of transfers to the agents be zero. In other words, there is no net transfer into or out of the economy.

Definition 4.1. A mechanism μ is *budget balanced* (BB) if for all θ, $\sum_{i=1}^{n} t_i(\theta) = 0$.

A weaker variant of budget balance is *feasibility* which allows a mechanism to accumulate a budget surplus but not a deficit. So long as the accumulated surplus can be disposed off elsewhere in the economy, this can be justified. Otherwise, it is an efficiency loss.

It is worth noting that there are mechanisms satisfying EFF and SP which run a budget surplus, an example of which is the well-known *pivotal mechanism*. In the queueing context, the pivotal mechanism serves everyone in the reverse (ascending) order of their waiting costs. Each agent pays the sum of waiting costs of those served behind him. This mechanism runs a budget surplus at all profiles.

Definition 4.2. A mechanism μ is *feasible* (F) if for all θ, $\sum_{i=1}^{n} t_i(\theta) \leq 0$.

We start with budget balance and show that none of the mechanisms characterized in Theorem 3.4 satisfies BB.

Proposition 4.3. There is no mechanism satisfying EFF, SP, EE and BB.

Proof. From Theorem 3.3, it follows that a mechanism (σ, t) satisfies EFF, SP and EE only if the transfers t_i satisfy (3.2). Applying BB and then simplifying it, we obtain that for all profiles θ,

$$\sum_{j \in N} (\sigma_j(\theta) - \sigma_0)\theta_j = \frac{nc}{n-1}. \tag{4.1}$$

We now have an impossibility as the left-hand side of (4.1) is dependent on θ (no matter how we choose σ_0) while the right-hand side is a constant. □

While budget balanced mechanisms are not possible, it turns out—rather unexpectedly—that there are feasible mechanisms satisfying EFF, SP and EE. The following result characterizes all such mechanisms. In particular, we show that $\sigma_0 = 1$ and $c \leq 0$.

Theorem 4.4. A mechanism μ satisfies EFF, SP, EE and F if and only if the transfers satisfy

$$\forall \theta, \forall i \in N: \; t_i(\theta) = \sum_{j \in N \setminus \{i\}} (1 - \sigma_j(\theta))\theta_j + c, \quad c \leq 0. \tag{4.2}$$

Proof. If μ satisfies (4.2), then $t_i(\theta) = -\sum_{j \in N\setminus\{i\}} (\sigma_j(\theta) - 1)\theta_j + c < 0$ for all $i \in N$ and hence F holds. From Theorem 3.3, it also follows that if a mechanism satisfies (4.2), then it satisfies EFF, SP and EE. Hence sufficiency of (4.2) is established.

To establish the necessity of (4.2), we only need to show the necessity of $c \leq 0$ and $\sigma_0 = 1$ using F. Suppose first that $c > 0$. Consider a profile θ such that $\theta_i = 2c/n(n-1)$ for all $i \in N$. Then, using (3.2) of Theorem 3.3, we get $\sum_{i \in N} t_i(\theta) = (2\sigma_0 - 1)c/2 > 0$ for any $\sigma_0 \in \{1, \ldots, n\}$, and hence we have a violation of F. Therefore, $c \leq 0$.

Next, suppose that $\sigma_0 = k$, $k \in \{2, \ldots, n\}$. Consider a profile θ such that $\theta_1 = \cdots = \theta_k = a > \theta_{k+1} = \cdots = \theta_n = b > 0$. For this profile,

$$\sum_{i \in N} t_i(\theta) = (n-1)\left[\frac{k(k-1)a}{2} - \frac{(n-k)(n-k+1)b}{2}\right] + nc. \tag{4.3}$$

Since $k > 1$ and $c \leq 0$, we can select

$$a = \frac{1}{k(k-1)}\left[3 + \frac{2n|c|}{n-1}\right] > b = \frac{2}{n(n-1)} \tag{4.4}$$

and then substituting these values in (4.3) and simplifying it, we get

$$\sum_{i \in N} t_i(\theta) = (n-1)\left[\frac{3}{2} - \frac{(n-k)(n-k+1)}{n(n-1)}\right] > 0. \tag{4.5}$$

This violates F showing that $\sigma_0 = 1$ is necessary. □

Remark 4.5. Let μ^* denote the set of mechanisms that satisfies EFF, SP, EE and F. From Theorem 4.4, it follows that if $\mu, \mu' \in \mu^*$, then for each profile θ, the difference between the transfer of any agent $i \in N$ *across the two mechanisms* μ *and* μ' is the (agent and profile independent common) constant c. Since this constant c is restricted to be non-positive, it follows that the mechanism $\mu \in \mu(c)$ that *minimizes the budget surplus* is the one for which $c = 0$. It is also clear from (4.2) of Theorem 4.4 that, unlike Yengin [29], one cannot place any upper bound on the budget surplus.

5. Impossibility Results

Theorem 3.4 shows that there is a class of mechanisms satisfying EFF, SP and EE. We now ask what happens if we impose two additional desirable properties: weak group strategyproofness and no envy. Unfortunately, we get negative results.

5.1. Weak Group Strategyproofness.

A mechanism is *weak group strategyproof* if there is no deviation by a group making all deviating members strictly better-off. Call two profiles θ and θ' *S-variants* if $\theta_i = \theta'_i$ for all $i \in N \backslash S$.

Definition 5.1. A mechanism (σ, t) is *weak group strategyproof* (WSP) if for all S-variants θ and θ', $u_i(\sigma_i(\theta), t_i(\theta); \theta_i) \geq u_i(\sigma_i(\theta'), t_i(\theta'); \theta_i)$ for at least one $i \in S$.

This concept has been used by, among others, Bogomolnaia and Moulin [3], Moulin and Shenker [19], Mutuswami [20], and Mitra and Mutuswami [17]. It is obvious that WSP implies SP.

Proposition 5.2. For $n \geq 3$, there is no mechanism satisfying EFF, WSP and EE.

Proof. Let (σ, t) be a mechanism satisfying EFF, WSP and EE. Since WSP implies SP, we can use Theorem 3.4 to infer that the reference bundle for any profile θ is $(\sigma_0, t_0(\theta))$ where $t_0(\theta) = \sum_{j \in N} (\sigma_0 - \sigma_j(\theta))\theta_j + c$. The allocation of agent i is $(\sigma_i(\theta), t_i(\theta))$ where $t_i(\theta) = \sum_{j \in N \setminus \{i\}} (\sigma_0 - \sigma_j(\theta))\theta_j + c$.

Consider profiles θ and θ' such that $\theta_1 > \cdots > \theta_n$ and, for all $i \in N$, $\theta'_i = \theta_i + x$, $x > 0$, $\theta'_i \in (\theta_i, \theta_{i-1})$ for all $i \in \{2, \ldots, n\}$. For these profiles, we show that there is a violation of WSP for all choices of σ_0.

(1) $\sigma_{\mathbf{0}} = \mathbf{1}$: Let $(\theta'_2, \theta'_3, \theta_{N \setminus \{2,3\}})$ be the true profile. We can check that agents 2 and 3 can profitably manipulate via the misreports θ_2 and θ_3.
(2) $\sigma_{\mathbf{0}} = \mathbf{n}$: Let θ be the true profile. Here, agents $n-2$ and $n-1$ can profitably manipulate via the misreports θ'_{n-1} and θ'_{n-2}.
(3) $\sigma_{\mathbf{0}} \neq \{\mathbf{1}, \mathbf{n}\}$: Let $\left(\theta'_{\sigma_0+1}, \theta_{N \setminus \{\sigma_0+1\}}\right)$ be the true profile. In this case agents $\sigma_0 - 1$ and $\sigma_0 + 1$ can manipulate profitably via the misreports θ'_{σ_0-1} and θ_{σ_0+1}. □

Remark 5.3. When $n = 2$, all mechanisms satisfying EFF, SP and EE are also WSP. In particular, putting $c = 0$ and $\sigma_0 = k$, $k = 1, 2$, gives us the *k-pivotal mechanisms* identified in Mitra and Mutuswami [17] and shown to be WGS. Clearly, adding a constant to all transfers preserves the WGS property. It follows from the earlier discussion that putting $\sigma_0 = 1$ and $c \leq 0$ gives us feasibility as well in this case.

5.2. **No-envy and egalitarian equivalence.** In this sub-section we make a detailed comparison between our results obtained using egalitarian equivalence and the results already obtained in the literature using the normative concept of *no-envy*. The idea of this comparison is to see the differing implications of the two equity criteria in the context of queueing models.

No-envy was introduced by Foley [10] and requires that no agent should end up with a higher utility by consuming what any other agent consumes.

Definition 5.4. A mechanism μ satisfies *no-envy* (NE) if for all θ and all $i, j \in N$, $u_i(\mu_i(\theta); \theta_i) \geq u_i(\mu_j(\theta); \theta_i)$.

Remark 5.5. It is straightforward to see that no-envy implies queue efficiency. If agent i is not to envy j at the profile θ, we must have

$$-(\sigma_i(\theta)-1)\theta_i+t_i(\theta)\geq -(\sigma_j(\theta)-1)\theta_i+t_j(\theta) \text{ or } t_i(\theta)-t_j(\theta)\geq(\sigma_i(\theta)-\sigma_j(\theta))\theta_i.$$

Combining this with a similar inequality from the condition that agent j should not envy agent i, we obtain

$$(\sigma_i(\theta)-\sigma_j(\theta))\theta_i\leq t_i(\theta)-t_j(\theta)\leq(\sigma_i(\theta)-\sigma_j(\theta))\theta_j. \tag{5.1}$$

The two conditions on the transfers are compatible only when $(\sigma_i(\theta)-\sigma_j(\theta))(\theta_i-\theta_j)\leq 0$. Hence, if $\theta_i>\theta_j$, then $\sigma_i(\theta)<\theta_j(\theta)$.[9] This is the condition for efficiency. Note that this result imposes no restrictions on the transfers.[10]

In contrast to no-envy, egalitarian equivalence imposes no restriction on the choice of the queue provided we have the freedom in choosing transfers. To see this, let θ be a profile and let $(\sigma_0(\theta), t_0(\theta))$ be the reference bundle. Suppose the mechanism chooses the queue σ. It is easy to verify that egalitarian equivalence will be satisfied if the transfers satisfy the following restrictions:

$$\forall i\in N, \forall\theta: t_i(\theta)=(\sigma_i-\sigma_0(\theta))\theta_i+t_0(\theta).$$

As a matter of fact, the difference between no-envy and egalitarian equivalence extends beyond their implications for queue efficiency. We show below that these concepts are incompatible when there are at least four agents. No additional assumption is needed for this result. A variant of this result has been established by Chun [4] who showed that the two equity notions are incompatible if BB are additionally imposed. He also showed that when there are two or three agents, NE and EE are compatible even if BB is additionally required. Here we strengthen Chun's [4] negative result by showing that the impossibility holds even if BB is dropped from the list.

We note that the incompatibility of no-envy and egalitarian equivalence for the problem of assignment of objects in a general quasi-linear framework with three

[9] Since this is a queue on a single machine, $\sigma_i(\theta)\neq\sigma_j(\theta)$.

[10] This is a strengthening of Chun [4]'s result which assumes that the transfers satisfy budget balance.

agents was established by Thomson [25] and Thomson [26] also established this incompatibility in the context of time division.[11]

Proposition 5.6. If $n \geq 4$, then NE and EE are incompatible.

Proof. Let μ be a mechanism satisfying NE and EE. Let θ be a profile such that $\theta_1 > \cdots > \theta_n$. By EE, there exists $(\sigma_0(\theta), t_0(\theta))$ such that $-(\sigma_i(\theta) - 1)\theta_i + t_i(\theta) = -(\sigma_0(\theta) - 1)\theta_i + t_0(\theta)$ for all $i \in N$. Rewriting this, we get

$$\forall i \in N : t_i(\theta) = (\sigma_i(\theta) - \sigma_0(\theta))\theta_i + t_0(\theta). \tag{5.2}$$

Choose two agents $i, i+1$. As shown in Remark 5.5, NE implies EFF and hence, $\sigma_{i+1}(\theta) = \sigma_i(\theta) + 1$. Using (5.1), it follows that $\theta_{i+1} \leq t_{i+1}(\theta) - t_i(\theta) \leq \theta_i$. Using (5.2), it follows that $\theta_{i+1} \leq (\sigma_{i+1}(\theta) - \sigma_0(\theta))\theta_{i+1} - (\sigma_i(\theta) - \sigma_0(\theta))\theta_i \leq \theta_i$. This implies that

$$0 \leq (\sigma_0(\theta) - \sigma_i(\theta))(\theta_i - \theta_{i+1}) \leq \theta_i - \theta_{i+1}. \tag{5.3}$$

Since $\theta_i > \theta_{i+1}$, it follows that $0 \leq \sigma_0(\theta) - \sigma_i(\theta) \leq 1$. Note that the selection of i and $i+1$ has been arbitrary. By choosing $i = 1$, we obtain $0 \leq \sigma_0(\theta) - 1 \leq 1$ which implies that $\sigma_0(\theta) \in \{1, 2\}$. By choosing $i = n-1$, we obtain $\sigma_0(\theta) \in \{n-1, n\}$. The two restrictions on $\sigma_0(\theta)$ are incompatible when $n \geq 4$. □

As observed before, Chun [4] has shown that NE and EE are compatible when $n \leq 3$ even if we also require BB. Here, we ask another question: Are NE, EE and SP compatible when $n \leq 3$? The answer is yes. When $n = 2$, the k-pivotal mechanisms are an example. We have already observed that when there are just two agents, these mechanisms satisfy EFF, SP and EE. Chun, Mitra and Mutuswami [6] show that they also satisfy NE.

When $n = 3$, the proof of Proposition 5.6 shows that we must have $\sigma_0 = 2$. Using (3.3), we can compute the transfers easily.[12] Assuming that the constant is zero, the transfers for a profile θ such that $\theta_1 > \theta_2 > \theta_3$ are:

$$t_1(\theta) = -\theta_3,\ t_2(\theta) = \theta_1 - \theta_3,\ t_3 = \theta_1.$$

[11] In our terminology, this impossibility result is established together with queue efficiency and budget balance.

[12] Since NE implies EFF, the queue is determined by the EFF criterion.

The corresponding utilities are

$$u_1 = -\theta_3,\ u_2 = \theta_1 - \theta_2 - \theta_3,\ u_3 = \theta_1 - 2\theta_3.$$

It is straightforward to check that no agent envies any other agent. Not surprisingly, this mechanism is not budget balanced.

In a heterogenous object assignment problem with the option that each agent may be assigned more than one object, it was shown by Yengin [30] that EFF, SP, EE and NE are compatible. Hence, the fact that in the queueing problem each agent is assigned only one queue position (one object) along with queue efficiency imposes significant structure to drive this incompatibility in comparison to the general model of assigning objects (by Yengin [30]).

6. Conclusions

Queueing models (single server, identical serving costs, quasi-linear preferences) constitute a very special environment in which it is possible to implement the "first-best" (queue efficiency, budget balance and implementation in dominant strategies). Indeed, in a very general heterogenous good allocation setting, Mitra and Sen [18] have argued that the structure of allocation problems for which one can obtain balanced VCG mechanisms must be queueing 'like.'

Since it is rare to find environments where budget balanced VCG mechanisms exist, one natural question that arises when we do find such an environment is whether such mechanisms also have nice equity properties. Two standard equity notions are *no envy* and *egalitarian equivalence*. The question of whether there are dominant strategy mechanisms satisfying no-envy has been addressed by Kayi and Ramaekers [14]. Here we deal with the issue of VCG mechanisms and egalitarian equivalence.

We show that the set of queue efficient, strategyproof and egalitarian equivalent mechanisms is non-empty and provide a complete characterization of this class. The fact that this class of mechanisms is non-empty illustrates the speciality of the queueing model. However, none of these mechanisms satisfy additional desirable properties like budget balance or immunity to group deviations. This is hardly surprising.

From a more general mechanism design perspective, asking for egalitarian equivalent VCG mechanisms which are also budget balanced or weak group strategyproof is indeed asking for too much. In particular, if we want egalitarian equivalence then strategyproofness reduces the degrees of freedom of the reference bundle substantially by making the queue position fixed for all profiles. Therefore, one should not be disheartened to find a negative result with such strong properties. One surprising result is that there are feasible VCG mechanisms that satisfy egalitarian equivalence.

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