University of Leicester

DEPARTMENT OF ECONOMICS

POLLUTION ABATEMENT IN A MODEL OF CAPITAL ACCUMULATION AND ENDOGENOUS LONGEVITY

Dimitrios Varvarigos, University of Leicester, UK

Working Paper No. 08/3
January 2008
Updated February 2008

Pollution Abatement in a Model of Capital Accumulation and Endogenous Longevity

Dimitrios Varvarigos[‡]

University of Leicester

This version: February, 2008

Abstract

The effort to reduce pollution entails economic benefits because improved environmental quality advances the health status of the population and reduces mortality. Yet, there are also economic costs accruing from this effort because activities towards environmental improvement require resources to be extracted away from capital investment. This paper examines the extent to which pollution abatement policies may, ultimately, increase or decrease income. This is done in the context of a dynamic general equilibrium model, in which the interactions of the dynamics between capital accumulation and environmental quality occur through the flow of pollution generated by economic activity and the beneficial effect of environmental quality on longevity.

JEL classification: O13; O41; Q56

Keywords: Pollution abatement; environmental quality; longevity; capital accumulation

[‡] **Address:** University of Leicester, Department of Economics, University Road, Leicester LE1 7RH, England. **Telephone: ++44 (0) 116 252 2184** **E-mail:** d.varvarigos@le.ac.uk

1 Introduction

Existing empirical evidence advocates a statistically positive correlation between longevity and economic activity. Barro and Sala-i-Martin (1996) estimate an annual increase of the growth rate by 1.4 percentage points, on average, as a result of an increase in life expectancy by 13 years. Chakraborty (2004) uses data to show a strong positive link between a country's per capita GDP and the estimated life expectancy, at birth, for its population.

Intuitively, the relationship between longevity and economic development seems to be by-directional. On the one hand, in countries with higher GDP per capita, people have more income at their disposal to get better nutrition or afford medical care and medicines, whereas governments can extract more revenues (e.g., from income taxation) to provide various essential health services at a national level. On the other hand, the reduction in mortality reduces the effective rate of time preference, since it mitigates the risk of not enjoying the future benefits accruing from current (either human or physical) capital investments – effectively, increasing their expected rate of return. These are exactly the issues raised in various theoretical studies which examine the joint determination of economic development and longevity prospects within unified analytical frameworks (e.g., Chakraborty, 2004; Chakraborty and Das, 2005; Blackburn and Cipriani, 2005; Tang and Zhang, 2007).

Insofar as the health status of a person is the single most important indicator of his/her life expectancy, it becomes evident that any factor expected to shape the health profile of the population should be seriously considered in the analytical study of economic growth/development. One such factor is related to environmental quality – i.e., the quality and purity of air and non-living natural resources (such as water, soil etc.) and the abundance of living natural resources (such as forests, fish etc.). Chemicals, toxins, smoke, radioactive substances and litter (all these, in many cases, by-products of economic activity) contaminate and erode the natural environment – either directly or indirectly – resulting in a profoundly adverse impact to the health status of people who are exposed to polluted environments.[1] The study of

[1] See Footnote 10 for references to specific studies which discuss and report evidence on the health effects of various types of environmental pollution and degradation.

Pimentel *et al* (1998) makes a strong case for this point: they estimate that, each year, roughly 40% of deaths worldwide can be attributed to factors related with environmental degradation.

Yet, despite all the evidence on the hazardous health impact of poor environmental conditions, as well as its staggering quantitative aspect in terms of human life loss, there is a surprising absence of analyses – within a significant body of literature that incorporate environmental issues in neoclassical growth models – that formally include the matter of health status while jointly analysing the evolution of economic activity and environmental quality.[2] To the best of my knowledge, only Jouvet *et al* (2007) have considered the analytics of this issue. Specifically, they construct a two-period overlapping generations model in which the stock of environmental pollution (whose flow is a by-product of aggregate production) generates both negative and positive externalities: the former occur because the length of the second period of a person's lifetime is a decreasing function of pollution; the latter occur because reduced longevity implies that less people occupy the (fixed) available land at any moment of time, thus increasing the available land per person – the measure of utility-enhancing environmental quality in their model. In this framework, they show that, under certain parameter configurations, taxation of both capital and private health spending may be optimal, since both types of taxes can partially improve the decentralised outcomes that are subject to the external effects of pollution.

The significant costs of poor environmental conditions, and their aforementioned repercussions on various aspects of the quality of life for a considerable number of people all over the world, brings forth the argument in favour of active policies designed to abate pollution (e.g., clean-up activities, encouragement for the introduction/implementation of 'cleaner' technologies and methods of production, recycling, more efficient use of natural resources etc.). In economic terms, such policies entail both costs and benefits: the former relate to the fact that, as any other policy, they require the extraction

[2] Formal analyses on the interactions between economic growth and the environment are provided by Tahvonen and Kuuluvainen (1991), John and Peccherino (1994), Bovenberg and Smulders (1995, 1996), Byrne (1997) and Agnani *et al* (2005) among others. A comprehensive review on the subject is provided by Brock and Taylor (2004).

of real, private sector resources (e.g., income, capital, labour) away from directly productive activities, like investment; the latter relate to the idea that the improved health for the population, and the reduction of mortality rates, will stimulate labour productivity, reduce the costs incurred by the public health system for the care of people who suffer from the conditions related to environmental degradation, and promote a widespread increase in saving.

These considerations form the basis for the analysis presented in this paper. Specifically, I seek to examine whether (costly) abatement policies (given their beneficial aspect on the natural environment) will, ultimately, result in a reduction or an increase in national income. In order to analyse this issue, I construct a two-period overlapping generations model, with saving and capital accumulation, in which the probability of survival to the second period of lifetime is an increasing function of environmental quality. The latter is modelled as a renewable resource that degrades on account of the pollution generated as a by-product of aggregate economic activity. First, I derive the steady state for capital and environmental quality and check how changes to some of the model's structural parameters affect the long-run equilibrium. Subsequently, I introduce a type of pollution abatement policy through which a lump sum tax on workers' income finances the public input that alleviates the negative externality of pollution on the environment. As expected, of course, this policy results in an increase of the long-run equilibrium level of environmental quality. Nevertheless, the effect of this policy on the long-run equilibrium for capital (and, therefore, income) are not clear – meaning that capital formation may also be higher in the presence of abatement policies. Hence, the overall effect on steady state income may depend on the relative strength of structural parameters.

The results of the model are related to the analysis on pollution abatement and growth by Smulders and Gradus (1996). They also study conditions under which pollution abatement policies may increase or decrease growth. Specifically, they find that abatement will benefit growth if the preference for lower pollution is relatively low, if the productivity of abatement technologies is high and if agents are patient enough. They do this, however, in a different context of a continuous-time economy with ongoing growth and pollution modelled as a flow that affects the productivity of the output sector. This analysis utilises an overlapping generations framework with explicit dynamics for environmental quality which affects capital accumulation, not through productivity but through longevity prospects and their effect on aggregate saving behaviour. As a result, it is able to identify an additional factor as important on whether abatement policies improve long-run income – i.e., the productivity of the health sector of the economy.

The rest of the paper is organised as follows: Section 2 describes the basic economic environment. In Section 3, I analyse the economy's dynamics in terms of capital accumulation and environmental quality, and in Section 4, I derive the steady state equilibrium. Section 5 introduces the pollution abatement policy and analyses its impact on the steady state equilibrium. In Section 6, I conclude.

2 The Economy

Consider an artificial economy in which time is discrete and indexed by $t = 0, 1, ...\infty$. This economy is inhabited by a constant population of agents that belong to overlapping generations and face a potential lifetime of two periods – youth and old age. The population of young agents is normalised to unity.

A young agent born at time t is endowed with one unit of labour which she supplies inelastically to firms in exchange for the market wage ω_t. She then decides how much to consume and how much to save for retirement, given that, when old, she does not have any labour endowment and, therefore, any alternative source of income other than the principal plus earnings from saving. The only way through which an agent can save is by depositing funds to a financial intermediary. Financial intermediaries transform the funds they receive into capital which they rent to firms at a cost of R_{t+1} per unit.[3] They are perfectly competitive and provide a gross rate of return r_{t+1} to their depositors. At the end of the first period of her lifetime, the agent gives birth

[3] I assume that the use of capital in production results in full depreciation of its (productive) value.

to an offspring whose rearing costs are incorporated in the consumption bundle of a young agent.[4]

One deviation of this model from the standard overlapping generations setting, due to Diamond (1965), is the idea that survival to old age is not certain. Instead, after the birth of her offspring, a realisation of a mortality shock determines whether an agent survives towards old age. Specifically, I assume that she will survive with probability $\pi_t \in (0,1)$, whereas with probability $1-\pi_t$ she dies prematurely and cannot enjoy any activities (mainly, consumption) when old.[5] As I shall explain in more detail later, longevity is an increasing function of environmental quality – denoted by e_t – which takes the form of a renewable resource that adjusts over time according to the pre-existing quality of the natural environment and the degree of pollution, denoted by p_t, which is generated by economic activity.

Given that only agents who survive are able to consume in both periods, their *ex post* utility is given by $(1-\chi)\ln c_t^t + \chi \ln c_{t+1}^t$, where c_i^j denotes consumption at period i of an agent born at period j, and $\chi \in (0,1)$ is the psychological weight on the utility derived from future consumption. In case the agent passes away prematurely, *ex post* utility is given by $(1-\chi)\ln c_t^t$. Consequently, an agent's *ex ante* (i.e., expected) lifetime utility is given by

$$(1-\chi)\ln c_t^t + \pi_t \chi \ln c_{t+1}^t, \tag{1}$$

which she maximises subject to the constraints for consumption during youth and old age. Denoting saving by s_t, these constraints are given by $c_t^t = \omega_t - s_t$ and $c_{t+1}^t = r_{t+1} s_t$ respectively.

Consumption goods are produced and supplied by perfectly competitive firms. These firms hire labour from the young, L_t, and capital from financial

[4] Although the idea of endogenous fertility is an interesting consideration, in this paper I abstract from issues relating to population dynamics in order to keep my analysis tightly focused on the interactions between capital accumulation and the quality of natural environment.

[5] In this respect, my analysis treats the idea of longevity similarly to Ehrlich and Lui (1991), Chakraborty (2004) and Zhang and Zhang (2005) among others. The assumption that the agent gives birth prior to the realisation of the mortality shock relates to the argument outlined in Footnote 2.

intermediaries, K_t and combine them to produce Y_t units of output according to a neoclassical technology $Y_t = F(K_t, L_t)$ with $F' > 0$ and $F'' < 0$ for both arguments. The production function is assumed to be homogeneous of degree one in capital and labour. Therefore, for the remaining analysis, I shall utilise its intensive (i.e., per capita or per worker) form

$$y_t = f(k_t), \tag{2}$$

where $y_t = Y_t / L_t$, $k_t = K_t / L_t$, $f(0) = 0$, $f' > 0$ and $f'' < 0$.[6]

3 Dynamics

The description of the economy's fundamentals can be utilised for the derivation of its dynamic equilibrium. This is characterised through

Definition 1. *Given* $k_0, e_0 > 0$, *the dynamic equilibrium of the economy is a sequence of quantities* $\left\{c_t^{t-1}, c_t^t, c_{t+1}^t, s_t, L_t, Y_t, y_t, \pi_t, p_t, K_t, k_t, e_t\right\}_{t=0}^{\infty}$ *and prices* $\left\{\omega_t, r_t, R_t\right\}_{t=0}^{\infty}$ *such that:*

(i) *Given* ω_t, π_t *and* r_t, *the quantities* c_t^t, c_{t+1}^t *and* s_t *solve the optimisation problem of an agent born at period* t, $\forall t \geq 0$;

(ii) *Given* ω_t *and* R_t, *firms choose quantities for* L_t *and* K_t *to maximise profits at period* t, $\forall t \geq 0$;

(iii) *The labour market clears every period, i.e.,* $L_t = 1$ $\forall t \geq 0$;

(iv) *The goods market clears every period, i.e.,* $Y_t = c_t^t + c_t^{t-1} + K_{t+1}$ $\forall t \geq 0$;

(v) *The financial market clears every period, i.e.,* $K_{t+1} = s_t$ $\forall t \geq 0$.

3.1 Capital Accumulation

The optimisation problem of an agent born at time t, leads to a solution for saving given by

[6] Of course, given that the population of young workers is normalised to unity and that they all supply a unit of labour, per capital and aggregate quantities will be indistinguishable in equilibrium.

$$s_t = \frac{\chi\pi_t}{1-\chi+\chi\pi_t}\omega_t\,. \tag{3}$$

Equation (3) indicates that the agent will devote a fraction of her labour earnings towards retirement income, by depositing it to an intermediary when young. If survival was certain (i.e., if $\pi_t = 1$) the agent would save a fraction equal to the weight she assigns to the utility accrued from old age consumption. However, the possibility of premature death induces the agent to devote a lower amount for retirement income and increase her consumption during youth. This is an important aspect captured by the fact that the saving rate varies with the longevity prospects of the agent.

Profit maximisation by output producing firms requires that the per unit costs of production inputs are equal to their respective marginal products. Recall that capital and labour payments are denoted by R_t and ω_t respectively. Also recall that, in equilibrium, $L_t = 1$. Then

$$R_t = f'(k_t)\,, \tag{4}$$

and

$$\omega_t = f(k_t) - k_t f'(k_t) \equiv \omega(k_t)\,, \tag{5}$$

where $\omega' = -kf'' > 0$.[7]

There are two conditions describing the equilibrium in the financial market. First of all, the equality between aggregate saving and aggregate investment requires

$$k_{t+1} = s_t\,, \tag{6}$$

since $K_{t+1} = k_{t+1}$. Second, the fact that the channelling of capital into firms is undertaken by financial intermediaries who operate under perfect competition means that these intermediaries derive zero economic profits from their activities. Thus, their costs (i.e., the total return to all surviving savers) must

[7] I assume $-f'' - kf''' < 0$ to ensure $\omega'' < 0$. Parametrically, this assumption holds (among other cases) with a Cobb-Douglas technology.

be equal to their revenues (i.e., the revenues they receive from firms who rent capital).[8] Consequently, using (6) we get

$$\pi_t r_{t+1} = R_{t+1}. \tag{7}$$

The equilibrium condition in (6) together with equations (3) and (5) imply that

$$k_{t+1} = \frac{\chi \pi_t}{1 - \chi + \chi \pi_t} \omega(k_t). \tag{8}$$

Equation (8) indicates that the dynamics of capital accumulation depend on the survival prospects of agents. In particular, an increase in the probability of survival stimulates aggregate saving and, therefore, promotes capital accumulation.[9]

Earlier I indicated that, for the purposes of the present analysis, a major factor in the determination of longevity is the quality of the natural environment. Next, I elaborate on this idea.

3.2 The Quality of the Natural Environment

It is widely documented that one of the important characteristics of human health status and, therefore, the prospect of longevity is manifested by the quality of the environment (i.e., the cleanliness of air, soil and water, the availability of natural resources such as forestry and other forms of plantation etc.).[10] Denoting the quality of the environment by e_t, I assume that the

---

[8] As in Chakraborty (2004) and Tang and Zhang (2007), I appeal to the idea of a perfect annuity market in which the young deposit their saving to a mutual fund which promises to provide retirement income, provided that the depositor survives to old age. Otherwise, the income of those who die is shared equally by surviving members of the mutual fund.

[9] Of course, these results are consistent with the economy-wide resource constraint. To see this, recall that every period only π_t agents survive to maturity. With this in mind, use $L_t = 1$ $K_{t+1} = k_{t+1}$, the per-period budget constraints and equations (6)-(7) to write $c_t^{t-1} + c_t^t + K_{t+1} = \pi_{t-1} r_t s_{t-1} + w_t - s_t + k_{t+1} = R_t k_t + w_t = R_t k_t + w_t L_t$. With a constant returns, neoclassical technology, we have $R_t k_t + w_t L_t = Y_t$.

[10] See Grigg (2004) and various analyses in Holget *et al* (1999) for extensive discussions and evidence on the hazardous health effects of environmental toxins and air pollution

probability of survival is an increasing function of environmental quality, according to

$$\pi_t = \Pi(e_t), \quad e \in [0, \overline{e}], \tag{9}$$

where $\Pi(0) = \underline{\pi} \geq 0$, $\Pi(\overline{e}) = \overline{\pi} \in (\underline{\pi}, 1)$, $\Pi' > 0$ and $\Pi'' < 0$. Equation (9) captures the idea that a cleaner and more prosperous natural environment promotes the health status of individuals and, therefore, increases their prospect for living longer. The parameter $\underline{\pi}$ captures the idea that there may be other (exogenous for this framework) indicators affecting health levels while the possibility that the quality of the environment reaches a certain maximum (i.e., denoted by $\overline{e}$) does not imply that survival becomes certain since other exogenous factors (e.g., accidents etc.) may still cause premature death.

Following Bovenberg and Smulders (1996), I treat the quality of the environment as a renewable resource which evolves according to

$$e_{t+1} = \gamma(e_t) - p_t, \tag{10}$$

with $0 < \gamma' < 1$ and $\gamma'' \leq 0$.[11] I also assume $\gamma(0) > 0$, which guarantees the existence of a non-negative solution for environmental quality, and $\gamma(\overline{e}) = \overline{e}$, which means that, in the absence of pollution, the steady state level for environmental quality would be at its maximum. The variable p_t captures the degradation of the natural environment due to pollution. I consider pollution

respectively. Koshal (1976) provide evidence on the negative impact of water pollution on health. Beckett *et al* (1998) show that trees act as biological filters that retain pollutant particles from the air. Raskin *et al* (2002) provide a comprehensive review on how various forms of plantation can be linked with health improvements, based on the idea that certain botanical extracts are important for the commercial production of medicines (such as antibodies and vaccines) by the pharmaceutical industry. McMichael (1993) estimates that the depletion of the stratospheric ozone layer (resulting from atmospheric pollution) by 1% leads to an increase of ultraviolet radiation – a major cause of skin cancer – by 1.4%.

[11] The restriction $\gamma' < 1$ follows Bovenberg and Smulders (1996) and guarantees the existence of a unique equilibrium. Allowing some range of values for which $\gamma' > 1$ would result in the existence of an additional steady state that would display the undesirable property of implying that greater pollution is beneficial for environmental quality. For this reason, I restrict my attention to $\gamma' < 1$ and rule out such an equilibrium.

as a by-product of economic activity.[12] Specifically, I assume that one unit of output produced generates $\tilde{\eta} > 0$ units of pollution. Therefore, after utilising the equilibrium condition $Y_t = y_t = f(k_t)$, we have

$$p_t = \tilde{\eta} f(k_t) . \tag{11}$$

Given the specification in (9), it is easy to check that the term $\chi\pi_t / (1 - \chi + \chi\pi_t)$ in equation (9) can be written as a function $\psi(e_t)$ such as $\psi' > 0$, $\psi'' < 0$, $\psi(0) = \chi\underline{\pi} / (1 - \chi + \chi\underline{\pi})$ and $\psi(\overline{e}) = \chi\overline{\pi} / (1 - \chi + \chi\overline{\pi})$. Therefore, (8) can be written as

$$k_{t+1} = \psi(e_t)\omega(k_t) \equiv g(k_t, e_t) , \tag{12}$$

while substitution of (11) in (10) yields

$$e_{t+1} = \gamma(e_t) - \tilde{\eta} f(k_t) \equiv h(k_t, e_t) . \tag{13}$$

Having described the fundamentals and the underlying relationships which depict the dynamics of the economy, the next step is to derive its steady state equilibrium. This is an issue to which I turn in the next Section.

4 The Long-Run Equilibrium

The dynamic equilibrium of the economy is described by the planar system of difference equations for physical capital and environmental quality given in (12) and (13) respectively. The solution to this system of equations is a steady state which can be characterised via

[12] Apart from the narrow notion of 'pollution' (e.g., toxic/chemical wastes, smoke, CO_2 emissions etc.), p_t could be broadened by incorporating the extraction of resources which are crucial for environmental quality (e.g., deforestation) – extraction which results from economic activity.

Definition 2. *The steady state equilibrium is a pair* $(\hat{k}, \hat{e})$ *such that* $\hat{k} = g(\hat{k}, \hat{e})$, $\hat{e} = h(\hat{k}, \hat{e})$.

Focusing our attention to the steady state solution, the first result can be derived in the form of

Lemma 1. *Define* $\phi(k) = k / \omega(k)$, *and assume* $\phi(0) = 0$ *and* $\phi' > 0$.[13] *Then, at the steady state, the dynamics of capital accumulation define a function* $k = \mu(e)$ *such that* $\mu' > 0$, *with* $\mu(0) = \underline{\mu} > 0$ *and* $\mu(\overline{e}) = \overline{\mu} > \underline{\mu}$.

Proof. Evaluate (12) at the steady state and rearrange to get $\phi(k) = \psi(e)$. Differentiation shows that $dk / de \equiv \mu_e(\cdot) = \psi_e(\cdot) / \phi_k(\cdot) > 0$. Furthermore, the conditions $\phi(0) = 0$ and $\phi' > 0$ imply that $\phi(\underline{k}) = \psi(0) = \chi\underline{\pi} / (1 - \chi + \chi\underline{\pi})$ and, therefore, define $\underline{k} \equiv \underline{\mu} = \phi^{(-1)}[\chi\underline{\pi} / (1 - \chi + \chi\underline{\pi})] \geq 0$. Similar analysis indicates that $\overline{k} \equiv \overline{\mu} = \phi^{(-1)}[\chi\overline{\pi} / (1 - \chi + \chi\overline{\pi})]$. Obviously, $\overline{\mu} > \underline{\mu}$ (and, therefore, $\overline{k} > \underline{k}$) by virtue of the function $\mu(\cdot)$ being monotonically increasing. ■

Intuitively, a better natural environment promotes longevity. Consequently, it induces agents to increase their saving. The latter effect stimulates capital accumulation due to higher investment. As a result, the dynamics of (12) generate a positive equilibrium relationship between the steady state levels of physical capital and environmental quality.

A second result, derived from the dynamics of equation (13), is stated as

Lemma 2. *At the steady state, the dynamics of environmental quality define a function* $k = v(e)$ *such that* $v' < 0$ *with* $v(0) = \overline{v} > 0$ *and* $v(\overline{e}) = 0$.

Proof. Evaluate (13) at the steady state and define $\delta(e) = \gamma(e) - e$ which is obviously a continuous function. Then $\tilde{\eta} f(k) = \delta(e)$. Differentiation shows that $dk / de \equiv v_e(\cdot) = \delta_e(\cdot) / \tilde{\eta} f_k(\cdot)$. Of course, $\delta_e(\cdot) = \gamma_e(\cdot) - 1$ is negative because

[13] These assumptions hold for a CES production function with relatively high elasticity of substitution between capital and labour (including the Cobb-Douglas case). Notice that the restriction $\phi' > 0$ corresponds to $\omega(k) - k\omega'(k) > 0$.

$\gamma_e < 1$ by assumption, therefore $dk / de < 0$. The condition $\gamma(\overline{e}) = \overline{e}$ implies that $\delta(\overline{e}) = 0$ in which case $f(k) = \delta(\overline{e}) / \tilde{\eta} = 0 \Rightarrow k = 0 = v(\overline{e})$. In addition, $\delta(e) \geq 0$ and $\delta' < 0$ imply that $\delta(0) = \delta^{MAX} > 0$, hence defining a positive capital stock $\breve{k} = f^{(-1)}(\delta^{MAX} / \tilde{\eta}) = v(0) = \overline{v}$. ■

When the level of capital investment increases, productive activity is stimulated. The resulting pollution erodes the natural environment and its resources to a greater extent. The process of regeneration occurs at a relatively slow rate – meaning that the environment cannot recuperate in full from the adverse impact of pollution. Consequently, the dynamics of equation (13) generate a negative relationship between the steady state levels of capital and environmental quality.

Prior to proceeding to the actual derivation and analysis of the steady state equilibrium, recall that $\mu' > 0$, $\mu(0) = \underline{\mu} > 0$ and $\mu(\overline{e}) = \overline{\mu} > \underline{\mu}$ (from Lemma 1) while $v' < 0$, $v(0) = \overline{v} > 0$ and $v(\overline{e}) = 0$ (from Lemma 2). As a result, if $\overline{v} < \underline{\mu}$ (i.e., $v(0) < \mu(0)$) then $v(e) < \mu(e)\ \forall e$ and, consequently, an interior equilibrium cannot exist. With this in mind, a meaningful result can be derived as

Proposition 1. *Assume* $v(0) > \mu(0)$. *Then, there exists a unique steady state equilibrium* $(\hat{k}, \hat{e})$ *such that* $\hat{k}, \hat{e} > 0$.

Proof. Given $v(0) > \mu(0)$, $\mu(\overline{e}) > v(\overline{e})$ (since $\overline{\mu} > 0$), $\mu' > 0$ and $v' < 0$, we conclude that there exists a some unique $\hat{e} \in (0, \overline{e})$ such that $\mu(\hat{e}) = v(\hat{e})$ with $\mu(e) < v(e)$ for $0 < e < \hat{e}$ and $\mu(e) > v(e)$ for $\hat{e} < e < \overline{e}$. Consequently, we can obtain $\hat{k} \in (0, \infty)$ such that $\hat{k} = \mu(\hat{e}) = v(\hat{e})$. ■

The existence of the interior steady state can allow safe conclusions and discussion concerning the interactions between capital accumulation and environmental quality only if this long-run equilibrium is non-trivial – that is, if the steady state satisfies the conditions for stability, whose notion is provided through

Definition 3. *The steady state* $(\hat{k}, \hat{e})$ *is locally stable (unstable) if the dynamics starting from any pair of initial values* (k_0, e_0), *in the neighbourhood of* $(\hat{k}, \hat{e})$, *satisfy* $k_\infty = \hat{k}$ *and* $e_\infty = \hat{e}$ ($k_\infty \neq \hat{k}$ *and/or* $e_\infty \neq \hat{e}$ *).*

In the Appendix, I show the configurations of parameter values that allow the equilibrium to be stable (a sink). Needless to say, for the remaining analysis I assume that the configuration that guarantees stability holds. In this respect it is meaningful to analyse how the long-run position for both capital investment and environmental quality will change in response of alterations in the model's structural parameters.

First, let us consider a scenario in which the economy experiences either an improvement in the prospects of longevity (e.g., an improvement in the technology for health services which may cause an increase of the survival probability for given values of environmental quality) or a decrease in the rate of time preference (i.e., an increase in χ) which makes agents less impatient to consume when young. Initially, the increase in saving will cause greater capital investment. Subsequently, for a given level of environmental quality, the capital stock will be higher. However, given the increase in pollution, the initial equilibrium state for the environment is no longer sustainable. Consequently, environmental quality will start declining and, as this happens, longevity prospects will be inhibited, causing a gradual decline in saving and investment – a decline, however, which is not strong enough to counter the initial increase in the capital stock. In the long-run, the economy will settle to a new equilibrium with a higher capital stock and lower environmental quality (Figure 1).

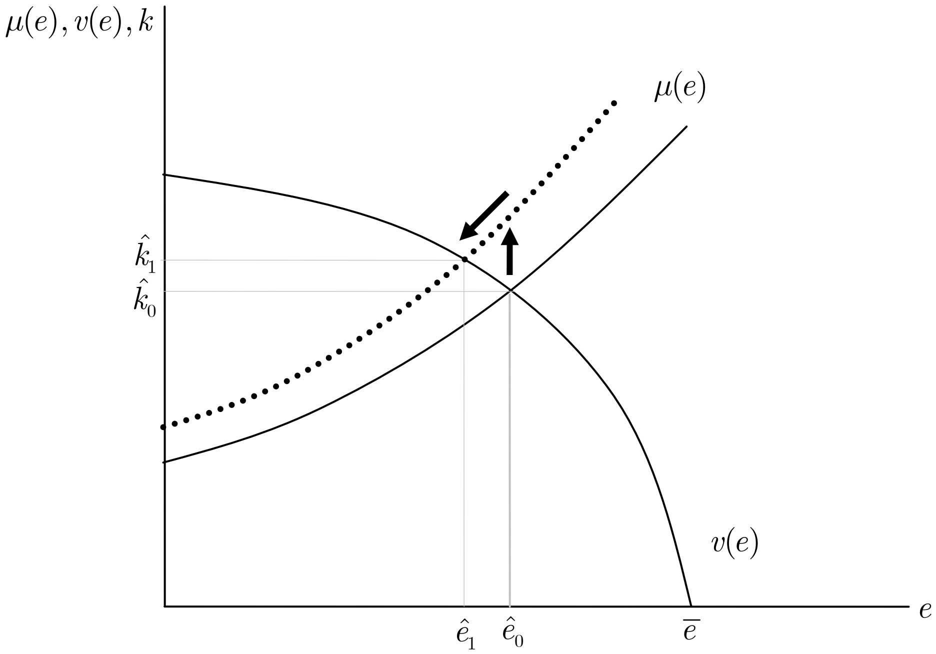

Figure 1. An increase in χ or π

Next, consider the scenario whereby the production technology emits more pollutant substances, requires the extraction of more natural resources and, therefore, erodes the environment to a greater extent for any given level of capital used in production (i.e., an increase in $\tilde{\eta}$). Initially, given the capital stock, the higher level of pollution causes a decline in environmental quality. As a result, the steady state equilibrium for capital, prior to the increase in $\tilde{\eta}$, is no longer sustainable. Gradually, the higher prospect of mortality leads to reduced saving and investment, and causes a gradual decline in the capital stock. Subsequently, production will decline as well, resulting in a decrease of the pollution caused by aggregate economic activity, albeit not to such an extent as to counter the increase in pollution due to the use of 'dirtier' technologies. Hence, the environmental stock will slightly improve although not enough as to account for the initial decline. In the long-run, the economy will settle to a new equilibrium with lower levels for both the capital stock and environmental quality (Figure 2).

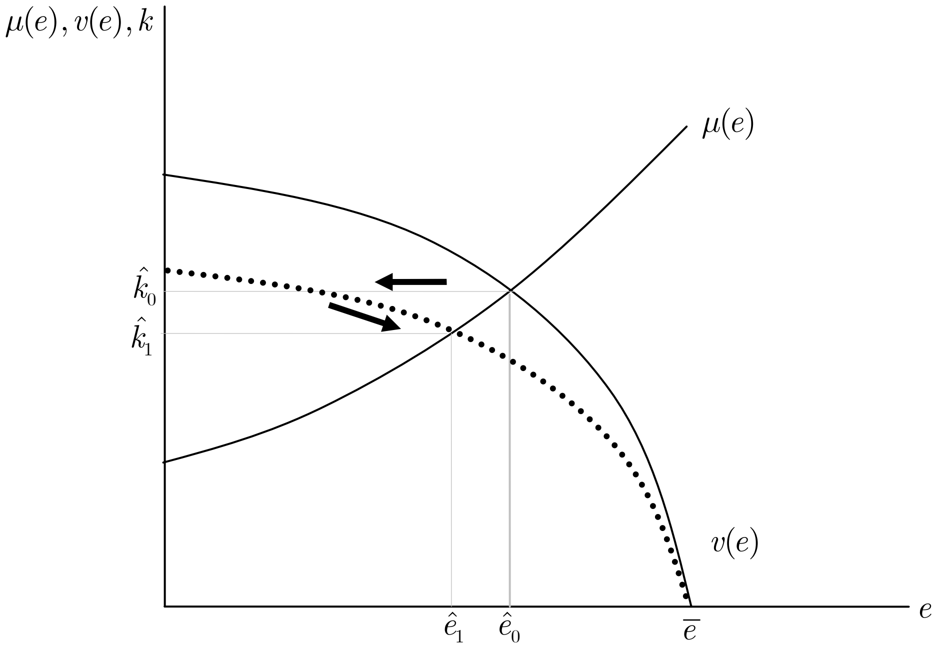

Figure 2. An increase in $\tilde{\eta}$

5 Pollution Abatement

The analysis presented towards the end of the previous Section indicated the beneficial aspects from reduced pollution for the long-run equilibrium of both the natural environment and the capital stock. A widely accepted view is that governments can mitigate the adverse effect of economic activity on the environment by implementing appropriate policies towards pollution abatement. Such policies may include, for example, various 'clean-up' activities; promotion of environmental R&D and adoption of environmentally friendlier technologies through appropriate incentives (e.g., subsidisation); construction/operation of recycling facilities; publicly funded campaigns (i) to raise awareness on environmental issues, and (ii) stimulate the mentality of recycling, buying environmentally-friendly goods etc.

Nevertheless, pollution abatement is not a costless process; it is an activity that requires specific resources to be allocated towards this purpose – resources that are diverted away from productive investment. The 'crowding-out' of private investment may eradicate the benefits of improved environmental conditions on capital formation and could, ultimately, impede the process of capital accumulation. The issue emerging is related to the relative strength of each isolated effect of pollution abatement on the equilibrium level of capital investment – an issue that will ultimately determine whether or not environmental policies entail a cost for the society in terms of lower national income in the long-run as many have suggested.

In terms of the present analysis, I shall consider the scenario whereby the government implements abatement policies that mitigate the adverse impact of aggregate economic activity on the environment. Specifically, I assume that the degree of pollution generated by economic activity is given by

$$p_t = \mathrm{H}(\alpha) f(k_t), \tag{14}$$

with $\mathrm{H}' < 0$, $\mathrm{H}'' > 0$, $\mathrm{H}(0) = \tilde{\eta}$ and $\lim_{\alpha \to \infty} \mathrm{H}(\alpha) = \underset{\sim}{\eta} > 0$ with $\tilde{\eta} > \underset{\sim}{\eta}$.

Equation (14) indicates that the amount of pollution and, therefore, the resulting degradation of the natural environment depend on the extent of abatement policies implemented by the government and signified by the presence of the fixed parameter α. This parameter captures the public input devoted towards various activities that inhibit the (adverse) environmental impact of the production process. When $\alpha = 0$, of course, the economy and its equilibrium are those described during the preceding analysis. As long as $\alpha > 0$, the quality of the environment is impeded to a lesser extent as a result of aggregate economic activity.

For the purposes of the current analysis, I assume that the government finances abatement policies by imposing a permanent, fixed lump-sum tax $\tau > 0$ on young workers and then utilises the total proceeds from taxation as to introduce its policy according to a continuously balanced budget, i.e., $\tau = \alpha \ \ \forall t$. Given this, it is straightforward to establish that the dynamics of capital formation and environmental quality now take the form

$$k_{t+1} = \psi(e_t)[\omega(k_t) - \alpha] \equiv \widehat{g}(k_t, e_t),$$
$$e_{t+1} = \gamma(e_t) - \mathrm{H}(\alpha) f(k_t) \equiv \widehat{h}(k_t, e_t).$$

In what follows, I shall employ a restriction on the fixed policy parameter. Specifically, I assume $\alpha < \omega(k_t) - k_t\omega'(k_t)$ which ensures that the disposable (i.e., after tax) income of young agents remains strictly positive.[14]

The remaining analyses focuses on the steady state. For $k_t = k$ and $e_t = e$, $\forall t$, the system of equations describing the long-run equilibrium becomes

$$k = \psi(e)[\omega(k) - \alpha] \equiv \widehat{g}(k,e)\,, \tag{15}$$

and

$$e = \gamma(e) - \mathrm{H}(\alpha)f(k) \equiv \widehat{h}(k,e)\,. \tag{16}$$

Given the above, the equilibrium is characterised through

Definition 4. *The steady state equilibrium is a pair* $(\widehat{k},\widehat{e})$ *such that* $\widehat{k} = \widehat{g}(\widehat{k},\widehat{e})$, $\widehat{e} = \widehat{h}(\widehat{k},\widehat{e})$ *and* $\tau = \alpha$ *with* $0 \leq \alpha < \omega(\widehat{k}) - \widehat{k}\omega'(\widehat{k})$.

Further analysis can yield

Lemma 3. *The equilibrium with pollution abatement policy,* $\alpha > 0$, *is described by two implicit functions* $k = \widehat{\mu}(e)$ *and* $k = \widehat{v}(e)$ *with* $\widehat{\mu}' > 0$ *and* $\widehat{v}' < 0$ *respectively. Compared with the scenario of no abatement policy,* $\alpha = 0$, *these functions satisfy* $\widehat{\mu}(e) < \mu(e)$ *and* $\widehat{v}(e) > v(e)$ $\forall e$.[15]

Proof. See the Appendix. ■

Now, we can establish

[14] Additionally, this restriction allows comparison of this scenario with the baseline model, given that it ensures that $k/[\omega(k) - \alpha]$ is monotonically increasing in k.

[15] The restriction $\alpha < \omega(\underset{\sim}{k}) - \underset{\sim}{k}\omega'(\underset{\sim}{k})$, where $\underset{\sim}{k} = \tilde{\mu}(0)$, applies.

Proposition 3. *There exists a unique steady state equilibrium* $(\widehat{k},\widehat{e})$ *such that* $\widehat{k},\widehat{e} > 0$.[16]

Proof. See the Appendix. ■

I shall address the issue of how abatement policies affect the long-run equilibrium outcomes for the environment and the capital stock by making a comparison of the equilibrium obtained in Section 4 (where $\alpha = 0$) against the outcomes that transpire for $\alpha > 0$. With respect to the quality of the environment, the result is expected and takes the form of

Proposition 4. *The implementation of abatement policy results in an unambiguous improvement for the quality of the environment, i.e.,* $\widehat{e} > \hat{e}$.

Proof. As shown in Proposition 1, with $\alpha = 0$ the equilibrium satisfies $\mu(\hat{e}) = v(\hat{e}) \Leftrightarrow \mu(\hat{e}) - v(\hat{e}) = 0$. Then, given Lemma 3, with $\alpha > 0$ we have $\widehat{\mu}(\hat{e}) - \widehat{v}(\hat{e}) < 0$. Observing that the difference $\widehat{\mu}(e) - \widehat{v}(e)$ is monotonically increasing in e we conclude that equilibrium can be established at some $\widehat{e} > \hat{e}$ such that $\widehat{\mu}(\widehat{e}) = \widehat{v}(\widehat{e})$. ■

With respect to the long-run level of capital investment (and, therefore, income) the impact of introducing abatement policies is established in

Proposition 5. *The implementation of abatement policy may lead to either a higher or a lower equilibrium level for capital in the long-run, i.e., either* $\widehat{k} > \hat{k}$ *or* $\widehat{k} < \hat{k}$.

Proof. Given that both functions $\widehat{\mu}(\cdot)$ and $\widehat{v}(\cdot)$ are continuous and monotonic, they have inverse functions, $\widehat{\mathrm{M}} = \widehat{\mu}^{(-1)}$ and $\widehat{\mathrm{N}} = \widehat{v}^{(-1)}$. Therefore, when $\alpha > 0$, $e = \widehat{\mathrm{M}}(k)$ and $e = \widehat{\mathrm{N}}(k)$ with $\widehat{\mathrm{M}}' > 0$ and $\widehat{\mathrm{N}}' < 0$ respectively. Of course, we can apply a similar reasoning when $\alpha = 0$, in which case both functions $\mu(\cdot)$ and $v(\cdot)$ are continuous and monotonic and they have inverse functions

[16] The stability condition, which can be modified (from the one provided in the Appendix) to account for the new equilibrium $(\widehat{k},\widehat{e})$ and $\alpha > 0$ is assumed to hold.

$\mathrm{M} = \mu^{(-1)}$ and $\mathrm{N} = v^{(-1)}$. Therefore, when $\alpha = 0$, $e = \mathrm{M}(k)$ and $e = \mathrm{N}(k)$ with $\mathrm{M}' > 0$ and $\mathrm{N}' < 0$ respectively. Recall that, in the absence of abatement policy, the equilibrium satisfies $\mathrm{M}(\hat{k}) = \mathrm{N}(\hat{k}) \Leftrightarrow \mathrm{M}(\hat{k}) - \mathrm{N}(\hat{k}) = 0$. It is straightforward to check that, because $\widehat{\phi}(\cdot)$ and $f(\cdot)$ are both increasing in k, comparison of the inverse functions reveals that $\widehat{\mathrm{M}}(k) > \mathrm{M}(k)$ and $\widehat{\mathrm{N}}(k) > \mathrm{N}(k)$ $\forall k$, as long as $\alpha > 0$. Consequently, we may either have $\widehat{\mathrm{M}}(\hat{k}) - \widehat{\mathrm{N}}(\hat{k}) > 0$ or $\widehat{\mathrm{M}}(\hat{k}) - \widehat{\mathrm{N}}(\hat{k}) < 0$. Now, observe that the difference $\widehat{\mathrm{M}}(k) - \widehat{\mathrm{N}}(k)$ is monotonically increasing in k. Hence, equilibrium can be established at some $\widehat{k} > \hat{k}$, if $\widehat{\mathrm{M}}(\hat{k}) - \widehat{\mathrm{N}}(\hat{k}) < 0$, or some $\widehat{k} < \hat{k}$, if $\widehat{\mathrm{M}}(\hat{k}) - \widehat{\mathrm{N}}(\hat{k}) > 0$, such that $\widehat{\mathrm{M}}(\widehat{k}) = \widehat{\mathrm{N}}(\widehat{k})$. ■

The upshot from the foregoing analysis is that – in addition to the improvement for the quality of the environment (which is expected) – pollution abatement policy, under certain cases, may also increase the long-run equilibrium for income, despite the fact that the government finances abatement policies by crowding out private investment. This happens if the improvement in environmental conditions promotes the health status of individuals to such an extent that the higher saving rate more than compensates for the loss of income in the process of capital formation. As Proposition 5 suggests, the increase in equilibrium income, following the implementation of pollution abatement activities, becomes more likely as long as the difference between the functions $\widehat{\mathrm{M}}(\cdot)$ and $\widehat{\mathrm{N}}(\cdot)$ is sufficiently low (i.e., if it is negative) for any given level of k.

The parameters that have a clear effect on this difference are those related to the 'health' technology and preferences. For example, the higher are the structural parameters of the technology behind $\Pi(\cdot)$ (e.g., more efficient health sector), and the higher is χ (e.g., less impatient consumers), then the lower is the difference $\widehat{\mathrm{M}}(\cdot) - \widehat{\mathrm{N}}(\cdot)$ for given k and the higher the likelihood that abatement policies – in addition to their beneficial effect on the environment – will allow the economy to achieve greater income.

6 Conclusion

Environmental quality is, possibly, the single most important determinant of people's longevity prospects. Pollution results in poor environmental conditions which demote the quality of life for a significant part of the world's population. It does this by being a major contributing factor of various diseases and, in many instances, even death. As such, the degradation of the environment – apart from the significant human and social welfare costs – may entail economic costs that take the form of lower labour productivity, significant reduction of the labour force and hindering of capital formation due to the scarcity of funds derived from economy-wide saving.

These considerations provide some support in favour of various activities – privately or publicly driven – that aim at improving the quality of the environment. Such activities mitigate the adverse (but, unfortunately, unavoidable) impact of economic activity on the environment – for example, by reducing the various wastes and emissions that pollute the environment and by allowing a more efficient use of natural resources. Of course, these activities are costly to initiate and implement – meaning that they necessarily extract real resources away from productive investment. Many have argued that, notwithstanding the obvious benefits to the natural environment, the economic costs of improving environmental quality may be higher than the economic benefits accruing from an improved environment. Should we acknowledge the view that improved environmental conditions have to come at an unavoidable cost in terms of lower income?

In this paper, I have tried to address this question. I built a simple overlapping generations model in which environmental quality promotes longevity by increasing the probability of survival towards old age – thus, encouraging saving behaviour during youth. Aggregate economic activity generates a negative externality since it causes pollution. Within this underlying framework, I derived equilibrium outcomes for capital intensity and environmental quality in two different scenarios – that is, with and without pollution abatement policies. Comparison of these two cases revealed that the steady state equilibrium for capital (and, consequently, income) could be either lower or higher in the presence of pollution abatement, with the latter case being more likely if (i) the underlying health technology is sufficiently advanced (i.e., relatively high values for the structural parameters affecting the probability of survival), and (ii) individuals' impatience to consume when young is sufficiently low.

Of course, there are other important issues that, while being left untouched in the present analysis, they could certainly enrich out understanding on the interactions between capital accumulation/growth and environmental quality under endogenous longevity. Such issues could relate to endogenous population dynamics, public and/or private health spending, private abatement activities, or even the distinction of different technologies according to the pollution they generate – and the optimal adoption of them by firms – to name but a few. This paper's framework abstracted from all these issues and was kept deliberately simple to ensure its tractability and its tight focus on the economic repercussions of policies towards pollution abatement. Undoubtedly, the addition of these issues could represent fruitful avenues for future research.

References

1. Agnani, B., Gutiérrez, M., and Iza, A. 2005. 'Growth in overlapping generations economies with non-renewable resources', *Journal of Environmental Economics and Management*, 50, 387-407
2. Azariadis, C. 1993. *Intertemporal Macroeconomics*. Blackwell. Cambridge (MA)
3. Barro, R. J., and Sala-i-Martin, X. 1996. *Economic Growth*. MIT Press. Cambridge (MA)
4. Beckett, K.P., Freer-Smith, P.H., and Taylor, G. 1998. 'Urban woodlands: their role in reducing the effect of particulate pollution', *Environmental Pollution*, 99, 347-360
5. Blackburn, K., and Cipriani, G.P. 2005. 'A model of longevity, fertility and growth', *Journal of Economic Dynamics and Control*, 26, 187-204
6. Bovenberg, A.L., and Smulders, S.A. 1995. 'Environmental quality and pollution-augmenting technological change in a two-sector endogenous growth model', *Journal of Public Economics*, 57, 369-391

7. Bovenberg, A.L., and Smulders, S.A. 1996. ‘Transitional impacts of environmental policy in an endogenous growth model’, *International Economic Review*, 37, 861-893
8. Brock, W.A., and Taylor, M.S. 2004. ‘Economic growth and the environment’, NBER working paper no. 10854
9. Byrne, M.M. 1997. ‘Is growth a dirty word? Pollution, abatement and endogenous growth’, *Journal of Development Economics*, 54, 261-284
10. Chakraborty, S. 2004. ‘Endogenous lifetime and economic growth’, *Journal of Economic Theory*, 116, 119-137
11. Chakraborty, S., and Das, M. 2005. ‘Mortality, human capital and persistent inequality’, *Journal of Economic Growth*, 10, 159-192
12. de la Croix, D., and Michel, P. 2003. *A Theory of Economic Growth: Dynamics and Policy in Overlapping Generations*. Cambridge University Press. Cambridge
13. Diamond, P.A. 1965. ‘National debt in a neoclassical growth model’, *American Economic Review*, 55, 1126-1150
14. Ehrlich, I., and Lui, F.T. 1991. ‘Intergenerational trade, longevity, and economic growth’, *Journal of Political Economy*, 99, 1029-1059
15. Grigg, J. 2004. ‘Environmental toxins: their impact on children’s health’, *Archives of Disease in Childhood*, 89, 244-250
16. Holgate, S.T., Samet, J.M., Koren, H.S., and Maynard, R.L., (eds.). 1999. *Air Pollution and Health*. Academic Press. London
17. John, A., and Peccherino, R. 1994. ‘An overlapping generations model of growth and the environment’, *Economic Journal*, 104, 1393-1410
18. Jouvet, P.A., Pestieau, P., and Ponthière, G. 2007. ‘Longevity and environmental quality in an OLG model’, Economix (Université Parix-X-Nanterre) working paper no. 2007-19
19. Koshal, R.K. 1976. ‘Water pollution and human health’, *Water, Air, and Soil Pollution*, 5, 289-297
20. McMichael, A.J. 1993. *Planetary Overload: Global Environmental Change and the Health of Human Species*. Cambridge University Press. Cambridge
21. Pimentel, D., Tort, M., D’Anna, L., Krawic, A., Berger, J., Rossman, J., Mugo, F., Doon, F., Shriberg, M., Howard, E., Lee, S., and Talbot,

J. 1998. ‘Ecology of increasing disease: population growth and environmental degradation’, *Bioscience*, 10, 817-826

22. Raskin, I., Ribnicky, D.M., Komarnytsky, S., Ilic, N., Poulev, A., Borisjuk, N., Brinker, A., Moreno, D.A., Ripoll, C., Yakoby, N., O’Neal, J.M., Corwell, T., Pastor, I., and Fridlender, B. 2002. ‘Plants and human health in the twenty-first century’, *Trends in Biotechnology*, 20, 522-531
23. Smulders, S., and Gradus, R. 1996. ‘Pollution abatement and long-term growth’, *European Journal of Political Economy*, 12, 505-532
24. Tahvonen, O., and Kuuluvainen, J. 1991. ‘Optimal growth with renewable resources and pollution’, *European Economic Review*, 35, 650-661
25. Tang, K.K., and Zhang, J. 2007. ‘Heath, education, and life cycle savings in the development process’, *Economic Inquiry*, 45, 615-630
26. Zhang, J., and Zhang, J. 2005. The effect of life expectancy on fertility, savings, schooling and economic growth: theory and evidence’, *Scandinavian Journal of Economics*, 107, 45-66

Appendix

Condition for stability of the steady state equilibrium $(\hat{k}, \hat{e})$. The Jacobian matrix associated with the system of equations (12) and (13) is

$$J = \begin{pmatrix} g_k(\hat{k}, \hat{e}) & g_e(\hat{k}, \hat{e}) \\ h_k(\hat{k}, \hat{e}) & h_e(\hat{k}, \hat{e}) \end{pmatrix},$$

where

$$g_k(\hat{k}, \hat{e}) = \psi(\hat{e})\omega'(\hat{k}) > 0\,,$$
$$g_e(\hat{k}, \hat{e}) = \psi'(\hat{e})\omega(\hat{k}) > 0\,,$$
$$h_k(\hat{k}, \hat{e}) = -\tilde{\eta} f'(\hat{k}) < 0\,,$$

and

$$h_e(\hat{k}, \hat{e}) = \gamma'(\hat{e}) > 0\,.$$

Denote the trace and the determinant of the Jacobian matrix by T and D respectively. Following a standard procedure (e.g., Azariadis, 1993; de la Croix and Michel, 2002), it is straightforward to show that the equilibrium $(\hat{k}, \hat{e})$ is locally stable (i.e., a sink) if $(1+D)^2 - T^2 > 0 \Leftrightarrow (1+D-T)(1+D+T) > 0$ and $|D| < 1$. Given the above, we have

$$T = g_k + h_e = \psi(\hat{e})\omega'(\hat{k}) + \gamma'(\hat{e}) > 0\,,$$

and

$$D = g_k h_e - g_e h_k = \psi(\hat{e})\omega'(\hat{k})\gamma'(\hat{e}) + \psi'(\hat{e})\omega(\hat{k})\tilde{\eta} f'(\hat{k}) > 0\,.$$

Obviously, $D + T + 1 > 0$. Next, we want to compute $D - T + 1$. We have

$$\begin{aligned} D - T + 1 &= g_k h_e - g_e h_k - g_k - h_e + 1 \\ &= g_k (h_e - 1) - g_e h_k - h_e + 1 \\ &= (1 - g_k)(1 - h_e) - g_e h_k \\ &= [1 - \gamma'(\hat{e})][1 - \psi(\hat{e})\omega'(\hat{k})] + \psi'(\hat{e})\omega(\hat{k})\tilde{\eta} f'(\hat{k}). \end{aligned}$$

By assumption, we have $0 < \gamma'(\hat{e}) < 1$ while, for a given $\hat{e}$, we have It is $\omega'(0) = \infty$ and $\omega'(\infty) = 0$. Given $\omega'' < 0$ then from (12), and for a given e, the steady state satisfies $0 < \omega'(\hat{k}) < 1$. As $\psi(\cdot) \in (0,1)$ $\forall e$ then $0 < 1 - \psi(\hat{e})\omega'(\hat{k}) < 1$ which means that $D - T + 1 > 0$. Therefore, the stability of the steady state requires that the condition $|D| < 1$ or, given that the determinant is positive, $D < 1$ holds. Thus, stability requires

$$\psi(\hat{e})\omega'(\hat{k})\gamma'(\hat{e}) + \psi'(\hat{e})\omega(\hat{k})\tilde{\eta} f'(\hat{k}) < 1$$

or, alternatively,

$$\psi'(\hat{e})\omega(\hat{k})\tilde{\eta} f'(\hat{k}) < 1 - \psi(\hat{e})\omega'(\hat{k})\gamma'(\hat{e}) \tag{A1}$$

Given that both equilibrium values $\hat{e}$ and $\hat{k}$ are determined by the model's structural parameters, then the condition in (A1) corresponds to a parameter restriction which I assume that holds. A casual indicates that an important requirement is that $\tilde{\eta}$ is sufficiently low. ■

Proof of Lemma 3. Define $\widehat{\phi}(k) = \dfrac{k}{\omega(k) - \alpha}$ which satisfies $\widehat{\phi}'(k) > 0$ given that $0 < \alpha < \omega(k) - k\omega'(k)$. Rearrange (15) to get $\widehat{\phi}(k) = \psi(e)$ which implicitly defines a function $k = \widehat{\mu}(e)$. Differentiation yields $dk / de \equiv \widehat{\mu}_e(\cdot) = \psi_e(\cdot) / \widehat{\phi}_k(\cdot) > 0$ therefore $\widehat{\mu}_e > 0$. In addition, notice that the presence of $\alpha > 0$ implies that $\widehat{\phi}(k) > \phi(k)$. For any given e (therefore, $\psi(e)$) equality is restored at lower values for k given that $\widehat{\phi}'(k) > 0$. Consequently, we conclude that $\widehat{\mu}(e) < \mu(e) \quad \forall e$. Next, define $\delta(e) = \gamma(e) - e$. Then $\mathrm{H}(\alpha) f(k) = \delta(e)$ which defines a function $\widehat{v}(e)$. Differentiation shows that $dk / de \equiv \widehat{v}_e(\cdot) = \delta_e(\cdot) / \mathrm{H}(\alpha) f_k(\cdot) < 0$, given $\delta_e(\cdot) = \gamma_e(\cdot) - 1 < 0$, therefore $\widehat{v}_e < 0$ and $\widehat{v}(\overline{e}) = 0$ because $\delta(\overline{e}) = \gamma(\overline{e}) - \overline{e} = 0$ by assumption. Furthermore, we can see that the presence of $\alpha > 0$ implies that $\mathrm{H}(\alpha) f(k) < \tilde{\eta} f(k)$. For any given e (therefore, $\delta(e)$) we can restore equality with higher values for k given that $f'(k) > 0$. As a result, $\widehat{v}(e) > v(e) \quad \forall e$. ■

Proof of Proposition 3. Once more, we can assume that $\widehat{v}(0) > \widehat{\mu}(0)$. Consequently, taking account that $\widehat{v}(\overline{e}) = 0 \Rightarrow \widehat{\mu}(\overline{e}) > \widehat{v}(\overline{e})$, $\widehat{\mu}' > 0$ and $\widehat{v}' < 0$, we conclude that there exists some unique $\widehat{e} \in (0, \overline{e})$ such that $\widehat{\mu}(\widehat{e}) = \widehat{v}(\widehat{e})$ with $\widehat{\mu}(\widehat{e}) < \widehat{v}(\widehat{e})$ for $0 < e < \widehat{e}$ and $\widehat{\mu}(\widehat{e}) = \widehat{v}(\widehat{e})$ for $\widehat{e} < e < \overline{e}$. Consequently, we can obtain $\widehat{k} \in (0, \infty)$ such that $\widehat{k} = \widehat{\mu}(\widehat{e}) = \widehat{v}(\widehat{e})$. ■