Imperfect Information

• The extensive-form games introduced last week all had **perfect information**.

• However, many economically interesting interactions have **imperfect information**.

• In this lecture, introduce extensive-form games of imperfect information...
  – Define such games, introduce **information sets** and redefine strategies appropriately.
  – Redefine subgames, and find Nash and subgame-perfect equilibria.
  – Identify a problem with subgame perfection: it doesn’t always rule out incredible threats.

• Briefly introduce an important class of games—**repeated games**.

• Important distinction between **finitely** and **infinitely** repeated games.
(Another) Dynamic Battle of the Sexes

Two friends have agreed to meet for coffee (but they’ve again forgotten at which cafe….) However, just before leaving home, the first friend thinks “hmmm…I could always stay in and do nothing instead…”

This is a combination of extensive-form and strategic-form elements.
Augmenting the Extensive Form

• Deal with these kinds of interactions by augmenting the extensive-form.

• **Information sets** are included to model a player’s ignorance of earlier moves...
Representing Strategic Forms as Extensive Forms

Represent the stag-hunt game in extensive form using information sets:

This is not the only representation. . . player 2 could move first, then player 1. Many extensive forms for a strategic form; unique strategic form for an extensive form.
Extensive-Form Games with Imperfect Information

Extensive-Form Games of Imperfect Information. These games have:

1. **Players.** As always, a set of players.

2. **Histories.** Exactly the same as before, including the “terminal” histories.

3. **Player Function.** As before, defined over non-terminal histories.

4. **Information.** An information set for $i$, written $I_i \in \mathcal{I}_i$, contains histories $h$ with $P(h) = i$, but where $i$ does not know which of $h \in I_i$ has occurred.

5. **Payoffs.** Payoffs for each player are defined over the terminal histories.

At any histories leading to a single information set $I_i$, player $i$ must have the same choices available—else $i$ would be able to deduce which history had occurred.
Applying the Definition to the Stag Hunt

Once again this looks like quite a complicated definition: but easy to apply.

- The set of players is not complicated! $N = \{1, 2\}$.

- $H = \{(\emptyset), (S), (R), (S, S), (S, R), (R, S), (R, R)\}$, and $Z = H \setminus \{(\emptyset), (S), (R)\}$.

- The player function is $P(\emptyset) = 1$, $P(S) = P(R) = 2$.

- There is a single information set for each player, $I_1 = \{\emptyset\}$, and $I_2 = \{(S), (R)\}$.

- Payoffs (assigned to the terminal histories) are e.g. $u_1(S, S) = u_2(S, S) = 5$. 
Strategies

A strategy for player $i$ tells player $i$ what to do at every $h$ where $P(h) = i$: but now player $i$ must choose the same action after any histories $h$ and $h'$ where $h, h' \in I_i$.

So $S$ is a strategy for player 2 and so is $R$, but “$S$ if $S$ and $R$ if $R$” is not. Player 2 must choose the same action (either $S$ or $R$) at every $h \in I_2 = \{(S), (R)\}$.
Representing Extensive Forms as Strategic Forms

With a definition for strategies in place, it is possible to represent any extensive-form game (of imperfect information) as a strategic-form game. For example:

![Diagram of an extensive-form game and its strategic-form representation]

Having done this, it is then straightforward to calculate Nash equilibria...
Mixed-Strategy Nash Equilibria

The pure equilibria are \((S, S)\) and \((\text{Stay In}, C)\). There are also many mixed...

\[
\begin{array}{c|cc}
 & S & C \\
\hline
\text{Stay In} & 0 & 0 \\
& 3.5 & 3.5 \\
S & 3 & 1 \\
& 4 & 1 \\
C & 0 & 4 \\
& 0 & 3 \\
\end{array}
\]

- \(C\) is dominated for player 1. \(C\) is now weakly dominated for player 2, so...

- If player 1 places probability \(p > 0\) on \(S\), player 2 plays \(S\) for sure.

- \(1\) plays \(\text{Stay In}\), and \(2\) puts \(q\) on \(S\). If \(3.5 \geq 4q + (1 - q) \iff q \leq \frac{5}{6}\) then equilibrium!
Subgames

A subgame is given when all but the histories that follow a singleton information set are removed, so long as this removal does not break any information set.
Subgame-Perfect Equilibrium

With subgames defined, once again a **subgame-perfect equilibrium** is a Nash equilibrium that induces Nash play in every subgame. Is this helpful?

No (proper) subgames! The only subgame is the whole game itself: all Nash are subgame perfect. Pure equilibria at \((A, b)\) and \((C, a)\) (mixed too). But is \(a\) credible?
An Incredible Subgame-Perfect Equilibrium

Player 2 sees if $C$ is not played. In this reduced game, $a$ is strictly dominated.

\[
\begin{array}{c|cc}
 & a & b \\
\hline
A & 1 & 2 \\
  & 0 & 2 \\
B & 2 & 4 \\
  & 1 & 0 \\
C & 3 & 3 \\
  & 1 & 1 \\
\end{array}
\]

- But this is not a subgame: it does not include all histories following $\emptyset$.
- Backward induction cannot be applied at all in this game.
- Thus, although $a$ is not credible, subgame perfection cannot rule it out.
Modifying the Extensive Form

But consider the following slight modification in the extensive form of this game:

\[ \begin{array}{c|c|c}
& a & b \\
\hline
DA & 1 & 2 \\
DB & 2 & 4 \\
CA & 3 & 3 \\
CB & 3 & 3 \\
\end{array} \]

The game after \( D \) is a subgame. So subgame perfection requires Nash play here. \((A, b)\) is Nash in the subgame \(\Rightarrow\) unique subgame-perfect equilibrium is \((DA, b)\).
A Modified Battle of the Sexes

Performing such modifications does not always generate a unique equilibrium.

(Pure) equilibria of subgame are \((S, S)\) and \((C, C)\). Equilibria of whole game are \(((\text{Out}, S), S)\), \(((\text{In}, S), C)\) and \(((\text{In}, C), C)\). Only first and last are subgame perfect.
A Horse-Like Game

Sometimes there’s no proper subgame and modification won’t help:

• There is a Nash equilibrium at \((A, a, r)\).

• There is a Nash equilibrium at \((D, a, \ell)\).

• But is there a problem with the second?

• Would player 2 actually play \(a\)?

• Only if player 3 was playing \(r\)!

All Nash are subgame perfect: it’s not enough to rule out incredible threats.
An Important Class of Games

- **Repeated games** are examples of imperfect-information extensive-form games.
  - These are games in which an identical (strategic-form) **stage game** is played many times.
  - Each time the players see what happened in the last round, before choosing their actions.
  - The stage game may be repeated **finitely** (for $T$ periods) or **infinitely** many times.

- Backward induction (and hence subgame perfection) is of some assistance...

- …in ruling out incredible threats (particularly in finitely repeated games).

- But the big problem is that virtually every threat becomes credible!

- Next lecture: the Nash and subgame-perfect equilibria of repeated games.