L’Hospital’s Rule

In several important cases, analysis of thermodynamic properties of solutions (and liquid mixtures) requires consideration of a term having the general form \(x \cdot \ln(x)\) where \(x\) is an intensive composition variable; e.g. molality, concentration or mole fraction. The accompanying analysis requires an answer to the question --- what value does the product \(x \cdot \ln(x)\) take in the limit that \(x\) tends to zero. But \(\lim (x \to 0) \ln(x) = -\infty\). The thermodynamic analysis has to take account of the answer to this question. In fact most accounts assume that the answer to the above question is ‘zero’. Confirmation that the latter statement is correct emerges from application of L’Hospital’s Rule (G. F. A. de l’Hospital, 1661-1704, marquis de Saint-Mesme). This rule allows the evaluation of terms having indeterminate forms. Most applications of this method usually involve the ratio of two terms each being a function of \(x\).

If \(f(x)/F(x)\) approaches either \([0/0]\) or \([\infty/\infty]\) when \(x\) approaches \(a\), and \(f'(x)/F'(x)\) [where \(f'(x)\) and \(F'(x)\) are first derivatives of \(f(x)\) and \(F(x)\)] approaches a limit as \(x\) approaches \(a\), then \(f(x)/F(x)\) approaches the same limit.

Example: If \(f(x) = x^2 - 1\) and \(F(x) = x - 1\)

Then \(\frac{f(x)}{F(x)} = \frac{x^2 - 1}{x - 1}\) and \(\frac{f'(x)}{F'(x)} = \frac{2 \cdot x^2}{1}\)

Then \(\lim (x \to 1) \frac{f'(x)}{F'(x)} = 2\)

Hence, \(\lim (x \to 1) \frac{f(x)}{F(x)} = 2\)

This rule can be proved using three assumptions.

(i) In the neighbourhood of \(x = a\), \(F(x) \neq 0\) if \(x \neq a\).

(ii) \(f(x)\) and \(F(x)\) are continuous in the neighbourhood of \(x = a\) except perhaps at \(a\).

(iii) \(f'(x)\) and \(F'(x)\) exist in some neighbourhood of \(x = a\) (except perhaps at \(x = a\)) and do not vanish simultaneously for \(x \neq a\).

In the present context the terms under consideration have a different form.

With reference to the term, \(x \cdot \ln(x)\),
\begin{align*}
f(x) &= \ln(x) \text{ and } F(x) = 1/x. \\
\text{Then } f'(x) &= 1/x \text{ and } F'(x) = -1/x^2. \\
\text{Hence, } f'(x)/F'(x) &= -x. \\
\text{Thus } \lim_{x \to 0} f'(x)/F'(x) &= 0. \\
\text{Hence, } \lim_{x \to 0} x \cdot \ln(x) &= 0.
\end{align*}