Multichannel Filter Banks and Non-dyadic Decompositions

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The M-band filter bank is a generalisation the two-channel filter bank in which the incoming signal is split into several components that lie in bands that cover the Nyquist frequency range, which is the interval $[0, \pi]$, measured in radians. (When the trigonometric functions are replaced by complex exponential functions, this becomes an interval of length 2π, which is usually taken to be $(-\pi, \pi]$.)

The components are extracted by an array of bandpass filters including a lowpass filter and a highpass filter at the top and at the bottom of the range. They are downsampled and encoded and then they are transmitted separately. At the receiving end, the components of the signal are decoded and upsampled before being smoothed by filters and recombined. The intention is that the recombined elements should reproduce the original signal exactly or as closely as possible.

It is most effective, in constructing battery of filters, to derive the bandpass filters and the highpass filter from a single lowpass prototype by the device of frequency shifting. It is also required to subsample the output of the bandpass filters. The next two sections prepare some of the groundwork by describing the methods of frequency shifting and downsampling.

Frequency Shifting

Let $x(t) = \{x_t; t = 0, \pm 1, \pm 2, \ldots\}$ and let $\xi(\omega) = \sum x_t e^{-\mathrm{i}\omega t}$ be the Fourier transform of the sequence $x(t)$. Then, the Fourier transform of the modulated sequence $x(t).\exp\{\mathrm{i}\gamma t\} = \{x_t e^{i\gamma t}\}$ is the function $\xi(\omega - \gamma) = \sum x_t e^{-\mathrm{i}(\omega-\gamma)t}$, which is to say that the effect of the exponential modulation is to shift the centre of the frequency-domain function from $\omega = 0$ to $\omega = \gamma$. Given that $z = e^{-\mathrm{i}\omega t}$, the z-transform of the original sequence can be written as $x(z) = \sum_t x_t z^t$ and that of the modulated sequence as $x(ze^{\mathrm{i}\gamma}) = \sum_t x_t (ze^{\mathrm{i}\gamma})^t$

Given that $\cos(\gamma t) = (e^{\mathrm{i}\gamma t} + e^{-\mathrm{i}\gamma t})/2$, it follows that the Fourier transform of $\cos(\gamma t).x(t)$ is $\xi(\omega - \gamma) + \xi(\omega + \gamma)$, and the result can be represented by writing

$$\cos(\gamma t).x(t) \longleftrightarrow \xi(\omega - \gamma) + \xi(\omega + \gamma). \tag{1}$$

This result provides the means of frequency shifting that is employed in generating a battery of bandpass filters from a single lowpass prototype filter $P(z)$. The object is achieved by multiplying the impulse response function of the lowpass filter by sinusoids at the frequencies corresponding to the central frequencies of the desired filters.

If it is desired to create a battery of M equally spaced filters covering the frequency range $[-\pi, \pi]$, then the central frequencies will be

$$\gamma_j = \pi(j + 1/2)/M; j = 0, 1, \ldots, M - 1. \tag{2}$$

Given a prototype filter $P(z)$ with a central frequency of zero and with a nominal passband on the interval $[-\pi/2M, \pi/2M]$, the jth filter can be given the form of

$$\begin{aligned} H_j(z) &= \alpha_j P(ze^{\mathrm{i}\gamma_j}) + \alpha_j^* P(ze^{-\mathrm{i}\gamma_j}) \\ &= \alpha_j P_j^+(z) + \alpha_j^* P_j^-(z) \\ &= H_j^+(z) + H_j^-(z), \end{aligned} \tag{3}$$

where $\alpha_j = \exp\{\mathrm{i}\theta_j\}$ is a term that effects a phase adjustment. The corresponding sequence of filter coefficients will be

$$h_j(t) = 2p(t).\cos\left(\frac{\pi}{M}\left\{j + \frac{1}{2}\right\}t + \theta_j\right). \tag{4}$$

Downsampling

Once the sequence $y(t)$ has been subject to a bandpass filter, it will be necessary to downsample the result in order to achieve a minimal representation of its contents. The subsampled sequence requires to be represented both in the time domain and in the frequency domain.

The time-domain representation is self-evident. Therefore, we may concentrate on frequency-domain representation. For this purpose, it is helpful to consider the case of downsampling by a factor of 3, before providing general formulae for the case of downsampling by a factor of M.

Consider the z-transform of the sequence $y(t)$, which is

$$y(z) = \sum_t y_t z^t = \cdots + y_0 + y_1 z + y_2 z^2 + y_3 z^3 + \cdots. \tag{5}$$

From this, the following three subseries can be derived:

$$\begin{aligned} y_0(z) &= \sum_t y_{3t} z^t &&= \cdots + y_0 + y_3 z + y_6 z^2 + y_9 z^3 + \cdots \\ y_1(z) &= \sum_t y_{3t+1} z^t &&= \cdots + y_1 + y_4 z + y_7 z^2 + y_{10} z^3 + \cdots. \\ y_2(z) &= \sum_t y_{3t+2} z^t &&= \cdots + y_2 + y_5 z + y_8 z^2 + y_{11} z^3 + \cdots \end{aligned} \tag{6}$$

The original series is expressed as a sum of the subseries as follows:

$$\begin{aligned} y(z) &= y_0(z^3) + zy_1(z^3) + z^2 y_2(z^3) \\ &= \sum_{j=0}^{2} z^j y_j(z^3) \end{aligned}. \tag{7}$$

We must also seek a way of expressing the downsampled subseries in terms original series $y(z)$. This can be achieved with the help of the delta function $\delta(t \bmod 3)$, which serves to eliminate the terms for which the index t is not a multiple of 3. Thus,

$$y_j(z) = \sum_t y_{t+j}\delta(t \bmod 3)z^{t/3}; \quad j = 0, 1, 2. \tag{8}$$

The delta function can be expressed as a sum of complex exponentials. In the case of $M = 3$, the essential complex exponential is $W_3 = \exp\{\mathrm{i}2\pi/3\}$. Then

$$\delta(t \bmod 3) = \frac{1}{3}\sum_{k=0}^{2} W_3^{kt} = \frac{1}{3}\sum_{k=0}^{2} e^{\mathrm{i}2\pi kt/3}, \tag{9}$$

To demonstrate the properties of this function, consider

$$\begin{aligned} W_3^0 &= \cos(0) + \sin(0) = 1, \\ W_3^1 &= e^{\mathrm{i}2\pi/3} = \cos(2\pi/3) + \mathrm{i}\sin(2\pi/3) \\ &= -1/2 + \mathrm{i}\sqrt{3/4}, \\ W_3^2 &= e^{-\mathrm{i}2\pi/3} = \cos(2\pi/3) - \mathrm{i}\sin(2\pi/3) \\ &= -1/2 - \mathrm{i}\sqrt{3/4}. \end{aligned} \tag{10}$$

On taking account of the three-point periodicity of W_3^Q, it can be seen that

$$\begin{aligned} &\frac{1}{3}\{(W_3^0)^0 + (W_3^1)^0 + (W_3^2)^0 = 1, \\ &\frac{1}{3}\{(W_3^0)^1 + (W_3^1)^1 + (W_3^2)^1 = 0, \\ &\frac{1}{3}\{(W_3^0)^2 + (W_3^1)^2 + (W_3^2)^2 = 0, \\ &\frac{1}{3}\{(W_3^0)^3 + (W_3^1)^3 + (W_3^2)^3 = 1. \end{aligned} \tag{11}$$

With increasing values of t, the cycle of $\sum_{k=0}^{2}(W_3^k)^t$ continues indefinitely to create the sequence $\{1, 0, 0, 1, 0, 0, 1, \ldots\}$.

Within the context of the z-transform, this sufficient to ensure that only the terms in successive powers of z^3 will be nonzero. Consider the addition of the power series $y(zW_3^0)$, $y(zW_3^1)$, and $y(zW_3^2)$, which may be arrayed as follows:

$$\begin{aligned} &\begin{matrix} \cdots + y_0(zW_3^0)^0 + \\ \cdots + y_0(zW_3^1)^0 + \\ \cdots + y_0(zW_3^2)^0 + \end{matrix} \begin{bmatrix} y_1(zW_3^0)^1 \\ y_1(zW_3^1)^1 \\ y_1(zW_3^2)^1 \end{bmatrix} \begin{matrix} + \\ + \\ + \end{matrix} \begin{bmatrix} y_2(zW_3^0)^2 \\ y_2(zW_3^1)^2 \\ y_2(zW_3^2)^2 \end{bmatrix} \begin{matrix} + y_3(zW_3^0)^3 + \\ + y_3(zW_3^1)^3 + \\ + y_3(zW_3^2)^3 + \end{matrix} \\ &\begin{bmatrix} y_4(zW_3^0)^4 \\ y_4(zW_3^1)^4 \\ y_4(zW_3^2)^4 \end{bmatrix} \begin{matrix} + \\ + \\ + \end{matrix} \begin{bmatrix} y_5(zW_3^0)^5 \\ y_5(zW_3^1)^5 \\ y_5(zW_3^2)^5 \end{bmatrix} \begin{matrix} + y_6(zW_3^0)^6 + \cdots \\ + y_6(zW_3^1)^6 + \cdots \\ + y_3(zW_3^2)^3 + \cdots \end{matrix} \end{aligned} \tag{12}$$

Here, the brackets surround the terms that will be eliminated in the addition of the three series. It is clear from (12) that

$$\begin{aligned} y_0(z^3) &= \frac{1}{3}\sum_{k=0}^{2}\sum_t y_t(W_3^k)^t z^t = \cdots + y_0 + y_3z^3 + y_6z^6 + y_9z^9 + \cdots \\ &= \frac{1}{3}\sum_{k=0}^{2} y(zW_3^k), \end{aligned} \tag{13}$$

where the final expression employs a functional notation for the z-transform. Hence

$$y_0(z) = \frac{1}{3}\sum_{k=0}^{2} y(z^{1/3}W_3^k). \tag{14}$$

More generally, when there is subsampling by a factor of M, we find that, with $W_M = \exp\{\mathrm{i}2\pi/M\}$ and $z = \exp\{-\mathrm{i}\omega\}$, the z-transform of the subsampled sequence can be written as

$$y_0(z) = \frac{1}{M}\sum_{k=0}^{M-1} y(z^{1/M}W_M^k) = \frac{1}{M}\sum_{k=0}^{M-1} y\left\{e^{-\mathrm{i}(\omega-2\pi k/M)}\right\}. \tag{15}$$

This shows that the discrete-time Fourier transform of the downsampled signal is the sum of M replicas of the spectrum of the original signal, displaced one from the next by $2\pi/M$ radians.

Polyphase Networks

An efficient implementation of an M-channel filter bank can be achieved by resorting to a polyphase decomposition of the data and the filters. The polyphase decomposition of a sequence indexed by $t \in \{0, \pm1, \pm2, \ldots\}$ creates M subsequences, indexed by $k = 0, 1, \ldots, M-1$, of which the elements of the kth subsequence bear the indices $qM + k$ with $q \in \{0, \pm1, \pm2, \ldots\}$

It will be helpful to consider a polyphase network that is devoid of data processing filters. Figure 1 portrays a network of three channels, which entails successive operations of time advances, downsampling, upsampling and time delays. Applied to the sequence $y(t) = \{\ldots, y_0, y_1, y_2, \ldots\}$, the operations of advancing and downsampling create the following three subsequences:

$$\begin{aligned} y(3t) &= \{\ldots,\ y_0,\ y_3,\ y_6, \ldots\}, \\ y(3t+1) &= \{\ldots,\ y_1,\ y_4,\ y_7, \ldots\}, \\ y(3t+2) &= \{\ldots,\ y_2,\ y_5,\ y_8, \ldots\}. \end{aligned} \tag{16}$$

Then, the operations of upsampling and delaying or lagging produce the following sequences:

$$\begin{aligned} y(3t \uparrow 3) &= \{\ldots,\ y_0,\ 0,\ 0,\ y_3,\ 0,\ 0, \ldots\}, \\ y([\{3t+1\} \uparrow 3] - 1) &= \{\ldots,\ 0,\ y_1,\ 0,\ 0,\ y_4,\ 0, \ldots\}, \\ y([\{3t+2\} \uparrow 3] - 2) &= \{\ldots,\ 0,\ 0,\ y_2,\ 0,\ 0,\ y_5, \ldots\}. \end{aligned} \tag{17}$$

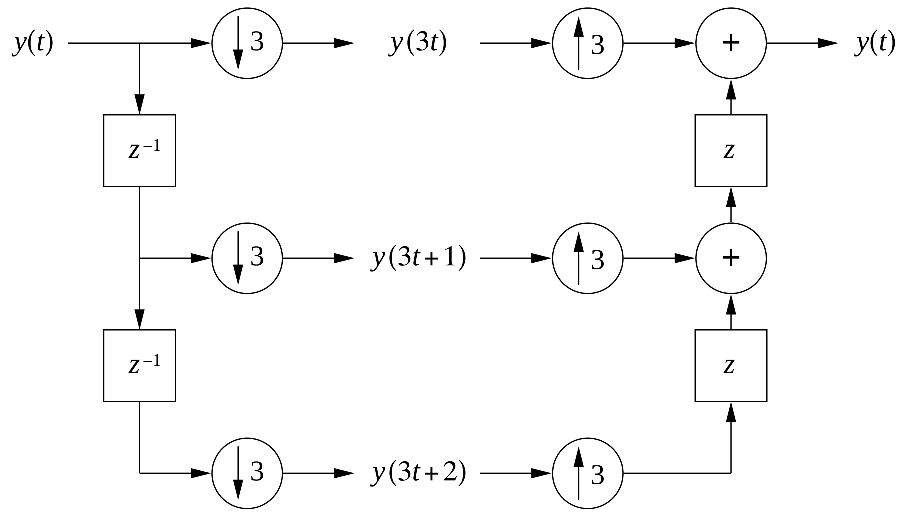

Figure 1. A three-channel serial to parallel converter that separates the data into three phases. On the input side, the operator z^{-1} advances the sequence, putting $y(t+1)$ in place of $y(t)$. On the output side the operator z imposes a delay.

By adding these three, the original data sequence can be recovered.

The presence of the time-advancing operations in this scheme implies either that the processing is applied off line to recorded data or else that there is a buffer on the input side that accumulates a sufficient number of elements and that entails at least as many time delays as the advances. If the processing is to be conducted in real time without an input buffer, then an alternative scheme is called for that entails only time delays. Such a scheme is portrayed in Figure 2.

Applied to the sequence $y(t+2)$, the operations of delaying and downsampling in this scheme create the following three subsequences:

$$\begin{aligned} y(3t+2) &= \{\ldots,\ y_2,\ \ y_5,\ \ y_8,\ldots\}, \\ y(3t+1) &= \{\ldots,\ y_1,\ \ y_4,\ \ y_7,\ldots\}, \\ y(3t) \quad &= \{\ldots,\ y_0,\ \ y_3,\ \ y_6,\ldots\}. \end{aligned} \tag{18}$$

Then, the operations of upsampling and delaying or lagging produce the following sequences:

$$\begin{aligned} y([\{3t+2\}\uparrow 3]-2) &= \{\ldots,\ 0,\ \ 0,\ \ y_2,\ \ 0,\ \ 0,\ \ y_5,\ldots\} \\ y([\{3t+1\}\uparrow 3]-1) &= \{\ldots,\ 0,\ \ y_1,\ \ 0,\ \ 0,\ \ y_4,\ \ 0,\ldots\}, \\ y(3t\uparrow 3) \qquad &= \{\ldots,\ y_0,\ \ 0,\ \ 0,\ \ y_3,\ \ 0,\ \ 0,\ldots\}. \end{aligned} \tag{19}$$

By adding them, the data sequence $y(t)$ can be recovered, which has a lag of two periods relative to the input sequence $y(t+2)$.

Filter Banks

Let the bandpass filters be denoted by $H_j(z); j=0,\ldots,M-1$, where bands of increasing frequency are indexed by increasing values of j, and let the corresponding

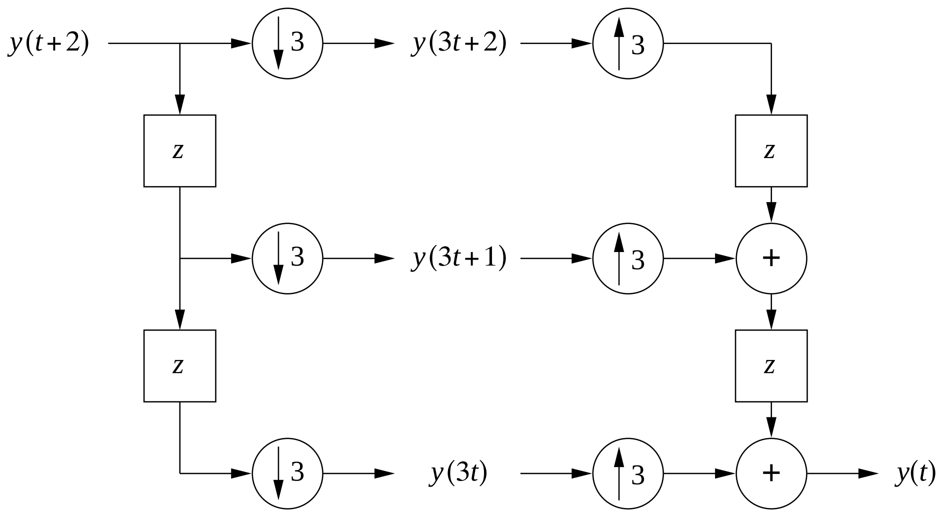

Figure 2. An alternative version of the three-channel serial to parallel converter that involves only time delays.

smoothing filters of the synthesis section be denoted by $F_j(z); j = 0, \ldots, M-1$. Then, in the absence of the upsampling and downsampling, the recombined signal would be denoted by

$$x(z) = \sum_{j=0}^{M-1} F_j(z)H_j(z)y(z). \tag{20}$$

In that case, a perfect reconstruction of the input signal $y(z)$ would be achieved if $\sum_j F_j(z)H_j(z) = I$.

With downsampling by a factor of M, the outputs of the analysis section of the jth channel of the filter bank become

$$\frac{1}{M}\sum_{k=0}^{M-1} H_j(z^{1/M}W_M^k)y(z^{1/M}W_M^k); \quad j = 0, \ldots, M-1, \tag{21}$$

wherein $W_M = \exp\{-\mathrm{i}2\pi/M\}$. Then, with upsampling by a factor of M and with smoothing, the output of the jth channel becomes

$$x_j(z) = \frac{1}{M}F_j(z)\sum_{k=0}^{M-1} H_j(zW_M^k)y(zW_M^k); \quad j = 0, \ldots, M-1. \tag{22}$$

Hereafter, for notational convenience, we shall, on occasion, omit the subscript from W_M. The outputs of the M channels are added to create the eventual output of the network:

$$\begin{aligned} x(z) &= \sum_{j=0}^{M-1} x_j(z) = \sum_{j=0}^{M-1}\frac{1}{M}F_j(z)\sum_{k=0}^{M-1} H_j(zW_M^k)y(zW_M^k) \\ &= \frac{1}{M}\left\{\sum_{j=0}^{M-1} F_j(z)H_j(z)\right\}y(z) + \frac{1}{M}\sum_{k=1}^{M-1}\sum_{j=0}^{M-1} F_j(z)H_j(zW^k)y(zW^k). \end{aligned} \tag{23}$$

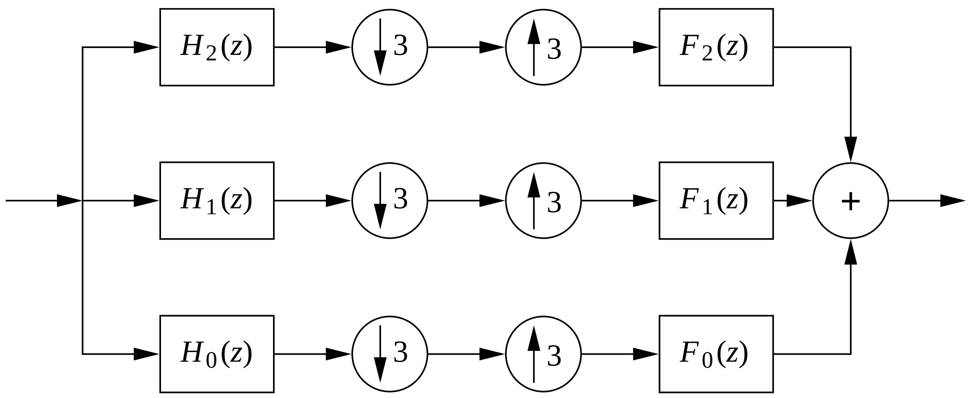

Figure 3. A depiction of the three-channel filter bank.

The second collection of the terms on the RHS is attributable to the effects of aliasing and imaging that are the results of downsampling and upsampling; and the objective is to minimise these effects, if not to eliminate them altogether. They will be eliminated, regardless of the value of the input $y(t)$, if

$$\sum_{j=0}^{M-1} F_j(z)H_j(zW^k) = 0, \quad \text{for} \quad k = 1, \ldots, M-1, \tag{24}$$

in which case, the input signal will be reconstucted perfectly if

$$\sum_{j=0}^{M-1} F_j(z)H_j(z) = M. \tag{25}$$

It may be helpful to represent the network via a matrix equation. Then, its output is represented by

$$x(z) = \frac{1}{M}\left[\, F_0(z) \quad F_1(z) \quad \ldots \quad F_{M-1}(z)\,\right] \times$$
$$\begin{bmatrix} H_0(z) & H_0(zW) & \ldots & H_0(zW^{M-1}) \\ H_1(z) & H_1(zW) & \ldots & H_1(zW^{M-1}) \\ \vdots & \vdots & & \vdots \\ H_{M-1}(z) & H_{M-1}(zW) & \ldots & H_{M-1}(zW^{M-1}) \end{bmatrix} \begin{bmatrix} y(z) \\ y(zW) \\ \vdots \\ y(zW^{M-1}) \end{bmatrix}. \tag{26}$$

It will be recognised that equation (26) is an evident generalisation of the case of $M = 2$ that has already been analysed in detail. In that case, W_M becomes $\exp\{i\pi\} = -1$ and equation (26) becomes

$$x(z) = \frac{1}{2}\left[\, F_0(z) \quad F_1(z)\,\right] \begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix} \begin{bmatrix} y(z) \\ y(-z) \end{bmatrix}. \tag{27}$$

In comparison with the former account, there has been a change of notation whereby $G(z) \longrightarrow H_0(z)$, $H(z) \longrightarrow H_1(z)$, $D(z) \longrightarrow F_0(z)$ and $E(z) \longrightarrow F_1(z)$.

The aliasing effects will be eliminated if

$$F_0(z)H_0(-z) + F_1(z)H_1(-z) = 0, \tag{28}$$

whereafter the signal will be reconstructed perfectly if

$$F_0(z)H_0(z) + F_1(z)H_1(z) = 2, \tag{29}$$

Pseudo Quadrature Mirror Filter Banks

If $h_j(t) \longleftrightarrow H_j(z)$ are the sequence of the coefficients of the jth filter and their Fourier transform, then the effects of downsampling and upsampling the sequence by a factor of M can be represented by writing

$$h_j\{(t \downarrow M) \uparrow M\} \longleftrightarrow \frac{1}{M}\sum_{k=0}^{M-1}\{H_j^+(zW^k) + H_j^-(zW^k)\}. \tag{30}$$

In demonstrating the pseudo quadrature mirror filters, one can rely on the case of $M = 3$. The generalisation to an arbitrary number of channels is straightforward.

In the case where $M = 3$ and where $j = 1$, which denotes the middle channel, the effect of downsampling and upsampling is to generate the following terms, which come from the RHS of (23):

$$H_1^+(zW) \quad \overrightarrow{H_1^-(z)} \quad H_1^+(zW^2) \quad H_1^-(zW) \quad \overrightarrow{H_1^+(z)} \quad H_1^-(zW^2) \tag{31}$$

These functions, each of which has a pass band with a nominal width of $\pi/3$ radians, jointly span the frequency interval $[-\pi, \pi]$.

The functions $H_1^-(z)$ and $H_1^+(z)$, which have nominal passbands on $[-2\pi/3, -\pi/3]$ and $[\pi/3, 2\pi/3]$, respectively, are rotated successively and in conjunction through angles of $2\pi/3$ radians in an anticlockwise direction, which is the direction of rising frequency, and which is indicated by the arrows.

The two succesive translations of $H_1^-(z)$, which is in the negative frequency range, will give rise to $H_1^-(zW)$ and $H_1^-(zW^2)$ which fall into the positive frequency range. The two succesive translations of $H_1^+(z)$, will give rise to $H_1^+zW)$ and $H_1^+(zW^2)$ which, in consequence of the circularity of the rotations, will be found in the negative frequency range.

An alternative procedure for generating the terms of (31) is to apply successive cosine modulations to the filter $H_1(z) = H_1^-(z) + H_1^+(z)$ to produce the following sequence:

$$H_1^+(zW) \quad \overleftarrow{H_1^-(z)} \quad H_1^+(zW^2) \quad H_1^-(zW^{-2}) \quad \overrightarrow{H_1^+(z)} \quad H_1^-(zW^{-1}). \tag{32}$$

The cosine modulations carry the positive-frequency and negative-frequency components of the filter in opposite directions.

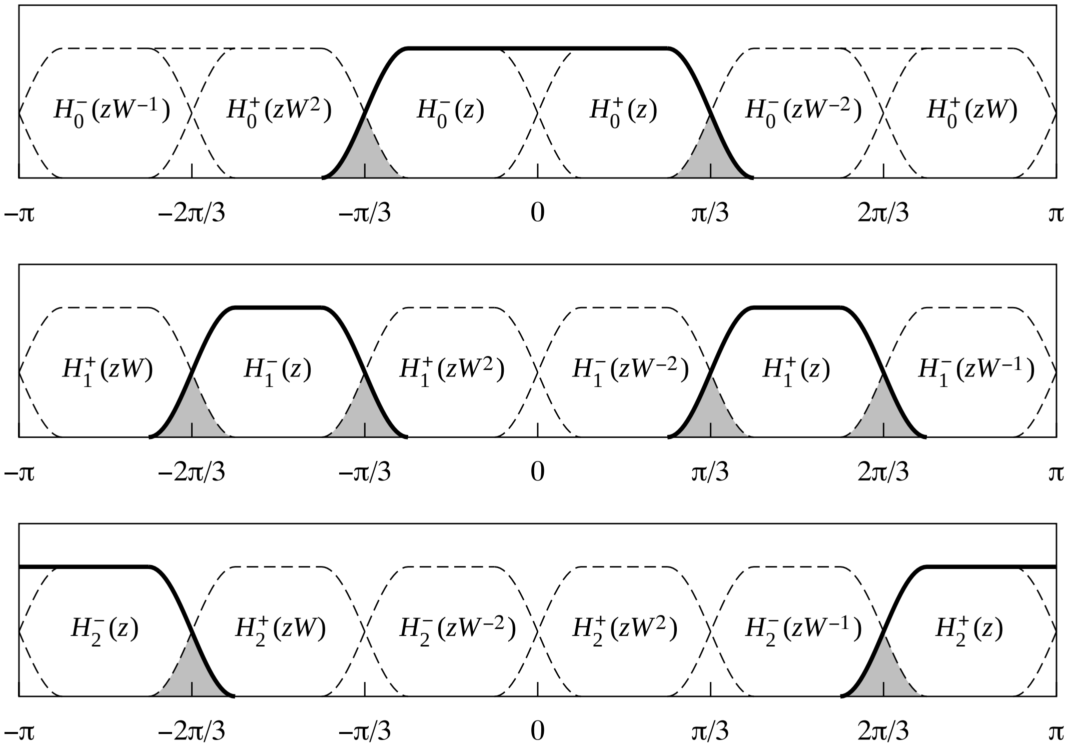

Figure 4. Aliasing cancellation is achieved by the interaction of the filters in adjacent channels. The figure depicts the squared gains of the filters.

Finally, one might choose to rotate the positive and negative frequency components of the filter in the clockwise and anticlockwise directions respectively:

$$H_1^+(zW^{-2}) \quad \overrightarrow{H_1^-(z)} \quad H_1^+(zW^{-1}) \quad H_1^-(zW) \quad \overleftarrow{H_1^+(z)} \quad H_1^-(zW^2)\,. \tag{33}$$

In fact, this is the most common choice in various expositions of the theory. (See, for example, Rothweiler 1983 and Vaidyanathan 1993.)

The three schemes can be reconciled in view of the fact that

$$H_1^{\pm}(zW^{-1}) = H_1^{\pm}(zW^2) \quad \text{and} \quad H_1^{\pm}(zW) = H_1^{\pm}(zW^{-2}), \tag{34}$$

where, for example, W^{-1} effects a clockwise rotation of $2\pi/3$ radians and W^2 effects an equivalent anticlockwise rotation of $4\pi/3$ radians, which carry the points on the unit circle to the same locations.

The advantage of the arrangements of (32) and (33) is that the positive and negative components of the filter images have a comparable notation, which enables them to be associated more readily and which allows us to adopt the following concise notation for the sum of associated components:

$$A_j(zW^{\pm k}) = H_j^-(zW^{-k}) + H_j^+(zW^k) \tag{35}$$

Figure 4 depicts the squared gains of the filters of the analysis stage together with their displaced images. The notation of the displacements corresponds to the cosine modulation scheme of (32). The figure reflects the assumption that only the frequency responses of adjacent filters are overlapping. This greatly reduces the number of nonzero terms in the equations of (24), arising from the aliasing and imaging, from a maximum of $M(M-1)$ to $2(M-1)$.

Given that the filters have finite supports in the frequency domain, it follows that the corresponding coefficients in the time domain must be infinite in number or else they must constitute a circular sequence. In either case, we may assume, for the present, that the sequences are symmetric about their central coefficients.

On setting $M = 3$, the equation of the network becomes

$$\begin{aligned}\{F_0(z)H_0(z) + F_1(z)H_1(z) + F_2(z)H_2(z)\}\\ + F_0(z)\{A_0(zW^{\pm 1}) + A_0(zW^{\pm 2})\}\\ + F_1(z)\{A_1(zW^{\pm 1}) + A_1(zW^{\pm 2})\}\\ + F_2(z)\{A_2(zW^{\pm 1}) + A_2(zW^{\pm 2})\}.\end{aligned} \tag{36}$$

We may assume that the synthesis filters $F_j(z)$ have same squared gain profiles as the analysis filters, which are the profiles marked by the bold lines in Figure 4. Then, it can be seen, in reference to the figure, that

$$F_0(z)A_0(zW^{\pm 1}) = 0 \qquad \text{and} \qquad F_2(z)A_2(zW^{\pm 2}) = 0. \tag{37}$$

It follows that perfect reconstruction will be achieved if

$$F_0(z)A_0(zW^{\pm 2}) = -F_1(z)A_1(zW^{\pm 2}) \quad \text{and} \tag{38}$$

$$F_1(z)A_1 zW^{\pm 1}) = -F_2(z)A_2(zW^{\pm 1}). \tag{39}$$

In that case, to overcome the effects of upsampling and downsampling, the aliasing effects in the vicinities of $\pm\pi/3$ within the first tranche of Figure 4 will need to be cancelled by those in the second tranche. The effects in the vicinities of $\pm 2\pi/3$ within the second tranche will will need to be cancelled by those is the third tranche.

The filters must be given sufficient flexibility to achieve the cancellations of (38) and (39). To this end, we may specify that

$$H_j(z) = H_j^-(z) + H_j^+(z) = \alpha_j^* P_j^-(z) + \alpha_j P_j^+(z), \tag{40}$$

$$F_j(z) = F_j^-(z) + F_j^+(z) = \alpha_j P_j^-(z) + \alpha_j^* P_j^+(z). \tag{41}$$

Here, $P_j^-(z) = P_0(z\exp\{-\mathrm{i}\gamma_j\})$ and $P_j^+(z) = P_0(\exp\{+\mathrm{i}\gamma_j\})$ are the components of the cosine modulated product of the symmetric prototype function $P_0(z) = P(z^{-1})$, and $\alpha_j = \exp\{+\mathrm{i}\theta_j\}$ and $\alpha_j^* = \exp\{-\mathrm{i}\theta_j\}$ are conjugate complex constants of unit magnitude that induce a phase displacement. It will be recognised that the symmetry of this specification ensures that the filter coefficients will be real-valued.

To determine the phase constants, it is sufficient to consider only the negative frequency components of the filters. In that case, the condition (38) for cancellation is that

$$\begin{aligned}0 =& F_0^-(z)H_0^+(zW^2) + F_1^-(z)H_1^+(zW^2)\\ =& \alpha_0 P_0^-(z)\alpha_0 P_0^+(zW^2) + \alpha_1 P_1^-(z)\alpha_1 P_1^+(zW^2).\end{aligned} \tag{42}$$

Here, we can set

$$P_0^+(zW^2) = P_1^-(z) \quad \text{and} \quad P_1^+(zW^2) = P_0^-(z), \tag{43}$$

so the condition for alias cancellation becomes

$$(\alpha_0^2 + \alpha_1^2)P_0^-(z)P_1^-(z) = 0, \tag{44}$$

which is satisfied if

$$\alpha_1^2 = -\alpha_0^2 \quad \text{or} \quad a_1 = \pm \mathrm{i}a_0. \tag{45}$$

By similar means, the condition of (39) can be rendered as

$$(\alpha_1^2 + \alpha_2^2)P_1^-(z)P_2^-(z) = 0, \tag{46}$$

which is satisfied if

$$\alpha_2^2 = -\alpha_1^2 \quad \text{or} \quad a_2 = \pm \mathrm{i}a_1. \tag{47}$$

The sequence of phase constants will be determined if one of its elements is known. Reference to equation (25) shows that, in the absence of aliasing effects, perfect reconstruction will be achieved if

$$\begin{aligned} M = \sum_{j=0}^{M-1} F_j(z)H_j(z) = \sum_{j=0}^{M-1} &\left\{|P_j^-(z)|^2 + |P_j^+(z)|^2\right\} \\ &+ (\alpha_0^2 + \alpha_0^{2*})P_0^-(z)P_0^+(z) \\ &+ (\alpha_{M-1}^2 + \alpha_{M-1}^{2*})P_{M-1}^-(z)P_{M-1}^+(z). \end{aligned} \tag{48}$$

Here, the second and the third terms arise because the components $P_j^-(z)$, $P_j^+(z)$ overlap when $j = 0, M-1$. These terms can be eliminated by setting

$$\alpha_0^4 = \alpha_{M-1}^4 = -1. \tag{49}$$

The solution is $\alpha_0 = \alpha_{M-1} = \exp\{\mathrm{i}\pi/4\}$. If we set

$$\alpha_j = \exp\{(-1^j)\mathrm{i}\pi/4\}, \tag{50}$$

then the conditions of (45) and (47) are also satisfied. The solution to the equation of (49) is represented in Figure 5.

The pseudo quadrature mirror filters are commonly implemented with a finite number of coefficients. In that case, it is not possible to restrict the frequency responses of the filters to limited interval within the frequency ranges. Therefore, the elimination of aliasing can only be achieved approximately; and one of the objectives in designing an appropriate prototype filter is to achieve a maximum attenuation within the stop bands.

If the filters are to be applied to the data in real time without an input buffer, then the sequences of their coefficients cannot be symmetric about a central element. In that case, the coefficients of the prototype filter will constitute a one-sided sequence $p(t) = \{p_0, p_1, \ldots, p_{N-1}\}$, with $p_t = p_{N-1-t}$.

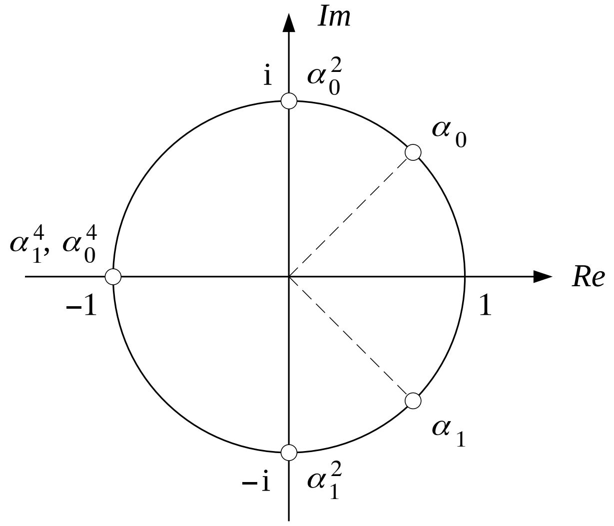

Figure 5. The phase constant $\alpha_0 = \exp\{i\pi/4\}$ is a solution to the equation $\alpha_0^4 = -1$.

With these restrictions, the coefficients of the jth filter of the analysis section will be represented by

$$h_j(t) = 2p(t).\cos\left(\frac{\pi}{M}\left\{j+\frac{1}{2}\right\}\left\{t+\frac{N-1}{2}\right\}+\theta_j\right), \tag{51}$$

whereas the coefficients of the jth filter of the synthesis section will be represented by

$$f_j(t) = 2p(t).\cos\left(\frac{\pi}{M}\left\{j+\frac{1}{2}\right\}\left\{t+\frac{N-1}{2}\right\}-\theta_j\right). \tag{52}$$

The Polyphase Formulation

The problem of perfect reconstructions can be approached via a polyphase analysis. The first stage is to expand each filter $H_j(z)$ into its M-channel polyphase representation:

$$H_j(z) = \sum_{k=0}^{M-1} z^k E_{jk}(z^M); \quad j = 0, 1, \ldots, M-1. \tag{53}$$

In matrix terms, this gives

$$\begin{bmatrix} H_0(z) \\ H_1(z) \\ \vdots \\ H_{M-1}(z) \end{bmatrix} = \begin{bmatrix} E_{0,0}(z^M) & E_{0,1}(z^M) & \ldots & E_{0,M-1}(z^M) \\ E_{1,0}(z^M) & E_{1,1}(z^M) & \ldots & E_{1,M-1}(z^M) \\ \vdots & \vdots & & \vdots \\ E_{M-1,0}(z^M) & E_{M-1,1}(z^M) & \ldots & E_{M-1,M-1}(z^M) \end{bmatrix} \begin{bmatrix} 1 \\ z \\ \vdots \\ z^{M-1} \end{bmatrix} \tag{54}$$

or

$$H(z) = E(z^M)e(z). \tag{55}$$

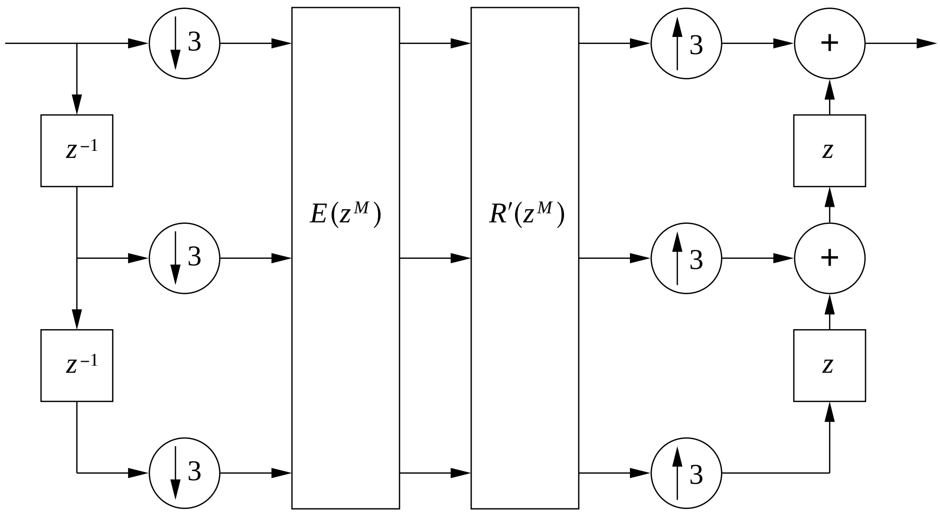

Figure 6. A three-channel filter bank can be constructed that separates the data into three phases. On the input side, the operator z^{-1} advances the sequence, putting $y(t+1)$ in place of $y(t)$. On the output side the operator z imposes a delay.

The synthesis filters can also be expanded via a polyphase decomposition:

$$F_j(z) = \sum_{k=0}^{M-1} z^{M-k-1} R_{kj}(z^M); \quad j = 0, 1, \ldots, M-1. \tag{56}$$

In matrix terms, this gives

$$\begin{bmatrix} F_0(z) \\ F_1(z) \\ \vdots \\ F_{M-1}(z) \end{bmatrix} = \begin{bmatrix} R_{0,0}(z^M) & R_{1,0}(z^M) & \ldots & R_{M-1,0}(z^M) \\ R_{0,1}(z^M) & R_{1,1}(z^M) & \ldots & R_{M-1,1}(z^M) \\ \vdots & \vdots & & \vdots \\ R_{0,M-1}(z^M) & R_{1,M-1}(z^M) & \ldots & R_{M-1,M-1}(z^M) \end{bmatrix} \begin{bmatrix} z^{M-1} \\ z^{M-2} \\ \vdots \\ 1 \end{bmatrix}, \tag{57}$$

or

$$f(z) = R(z^M)e^J(z). \tag{58}$$

An illustration of the polyphase filter bank is provided in Figure 6 for the case of $M = 3$. In will be seen that in the absence of the operators $E(z^M)$ and $R(z^M)$ the output would equal the input without any delay. It follows that, with the operators in place, a perfect reconstruction of the input signal would be achieved if

$$R'(z^M)E(z^M) = I. \tag{59}$$

A relationship can be established between the matrix of (26) and the polyphase matrix of (54). The polyphase expansion of the generic term within (26) is

$$\begin{aligned} H_j(zW_M^k) &= E_{j0}(\{zW_M^k\}^M) + \{zW_M\}E_{j1}(\{zW_M^k\}^M) + \\ &\qquad \cdots + \{zW_M^k\}^{M-1}E_{j(M-1)}(\{zW_M^k\}^M) \\ &= E_{j0}(z^M) + \{zW_M^k\}E_{j1}(z^M) + \\ &\qquad \cdots + \{zW_M^k\}^{M-1}E_{j(M-1)}(z^M). \end{aligned} \tag{60}$$

This is a matter of replacing z in equation (53) by zW_M The simplification of the second equality arises from the fact that $W_M^{jM} = 1$. Therefore,

$$\begin{aligned}&\begin{bmatrix} H_0(z) & H_0(zW) & \ldots & H_0(zW^{M-1}) \\ H_1(z) & H_1(zW) & \ldots & H_1(zW^{M-1}) \\ \vdots & \vdots & & \vdots \\ H_{M-1}(z) & H_{M-1}(zW) & \ldots & H_{M-1}(zW^{M-1}) \end{bmatrix} \\ &= \begin{bmatrix} E_{0,0}(z^M) & E_{0,1}(z^M) & \ldots & E_{0,M-1}(z^M) \\ E_{1,0}(z^M) & E_{1,1}(z^M) & \ldots & E_{1,N-1}(z^M) \\ \vdots & \vdots & & \vdots \\ E_{N-1,0}(z^M) & E_{N-1,1}(z^M) & \ldots & E_{M-1,M-1}(z^M) \end{bmatrix} \\ &\quad \times \begin{bmatrix} 1 & 1 & \cdots & 1 \\ z & zW & \cdots & zW^{M-1} \\ \vdots & \vdots & & \\ z^{M-1} & z^{M-1}W^{M-1} & \ldots & z^{M-1}W^{(M-1)^2} \end{bmatrix}.\end{aligned} \tag{61}$$

The matrix on the RHS may be factorised as the product of the matrix $D(z) = \text{diag}\{1, z, \ldots, z^{M-1}\}$ and the matrix of the Fourier transform $W(z)$ of order M. Thus, equation (40) may be written in summary notation as

$$H(z) = E(z^M)D(z)W. \tag{62}$$

The inverse mapping from $H(z)$ to $E(z^M)$ is $E(z^M) = H(z)\bar{W}D(z^{-1})$, which is a generalisation of that of (5.27).

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